Symplectic Geometry and Lefschetz Fibrations

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Abstract

This article is a survey on the classification of Lefschetz fibrations and its relation with symplectic geometry. Donaldson showed that any closed symplectic 4-manifold admits a Lefschetz pencil which can be blown up to give a Lefschetz fibration ([D1]). On the other hand, Gompf showed that any Lefschetz fibration having fibers with non-trivial homology class admits a symplectic structure ([GS]). The two results coupled to give a topological characterization of closed symplectic 4-manifolds.

Some works on the classification of Lefschetz fibrations are also sketched, from the classical ones such as handlebody decomposition of Lefschetz fibrations ([K]) and the complete classification in the genus 1 case ([Mo]), to the more recent ones like fiber sum stabilization of Lefschetz fibrations for any genus ([A1]) and the relationship between hyperelliptic Lefschetz fibrations and branched covers. ([ST1]).

摘要

本論文概觀 Lefschetz 纖維化之分類以及其與辛幾何之關係。Donaldson 證明了所有的閉四維辛流形均可有 Lefschetz 束，而 Lefschetz 束可於基點拉開而成爲 Lefschetz 纖維化 ([D1])。另一方面，Gompf 證明了所有容有非平凡同調纖維之 Lefschetz 纖維化或 Lefschetz 束之閉四維流形均爲辛流形 ([GS])。這兩個結果結合起來，則成了閉四維辛流形的拓撲特徵化。

本文亦概述 Lefschetz 纖維化分類的一些結果，其中包括比較經典的 Lefschetz 纖維化之柄體分解 ([K])、虧格爲 1 的 Lefschetz 纖維化的完滿分類 ([Mo])，以及比較近期的一般虧格的 Lefschetz 纖維化之纖維和穩定化 ([A1])、超橢圓 Lefschetz 纖維化與分枝覆蓋之關係 ([ST1])。

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Contents

Chapter 1

Introduction

The story starts at classifying simply connected 4-manifolds. The earliest examples of 4-manifolds are complex surfaces, and the easiest way to generate such examples is to consider zero sets of polynomials in $\mathbb{CP}^n$, which we call subvarieties of $\mathbb{CP}^n$ or algebraic surfaces. However, this only forms a limited class of 4-manifolds, and it is natural to seek generalization. It was well-known that every simply connected complex manifold is algebraic, and every algebraic manifold has an induced Kähler structure, i.e. a symplectic structure compatible with the complex structure, from $\mathbb{CP}^n$. Therefore it is of interest to construct examples of symplectic 4-manifolds.

It was shown long ago that a 4-manifold which is open or has non-empty boundary admits a symplectic structure if and only if it admits an almost complex structure ([McS], p.238). Therefore the classification of symplectic 4-manifolds is actually harder for closed manifolds (i.e. compact without boundary). Indeed, there are obstructions for a 4-manifold to be symplectic which occur only for closed manifolds (Section 2.2). On the positive side, symplectic 4-manifolds can be constructed by some purely topological information; Thurston showed that a bundle structure over a compact oriented surface (with an extra homological condition) suffices, and Gompf generalized the construction to cover Lefschetz fibrations and Lefschetz pencils, where the latter can be blown up to obtain the former (Section 2.3). Conversely, Donaldson showed that all symplectic 4-manifolds admit Lefschetz pencils (Section 2.4). These two results together provide a topological characterization of symplectic manifolds, and added motivation of studying Lefschetz fibrations as a topological object.

Lefschetz pencils and Lefschetz fibrations are initially studied by Lefschetz in order to understand the topology of complex manifolds. Then Kas' work in early 80's, which is now seen as classical, gave a fundamental way of studying Lefschetz fibrations by regarding them as handlebodies with handles attached in a way that can be characterized in terms of a finite sequence of positive Dehn twists of its generic fiber Σ_g, a compact oriented surface of genus g (Section 3.2). From this point of view, the classification of Lefschetz fibrations is reduced to a purely algebraic problem of classifying finite sequences of positive Dehn twists on Σ_g under a certain equivalence relation.

In general, the complexity of classification of Lefschetz fibrations increases rapidly with g, following the same trend as shown in $\mathrm{MCG}(\Sigma_g)$, the mapping class group of Σ_g. Also in early 80's, Moishezon completely solved the case $g = 1$, where $\mathrm{MCG}(\Sigma_1)$ is just $\mathrm{SL}_2(\mathbb{Z})$, by reducing the problem to a problem in matrix theory which was then solved by a theorem of Livne (Section 3.3). Then, in the recent decade, after Donaldson's and Gompf's results, Siebert and Tian showed that any genus 2 Lefschetz fibrations without reducible fibers and "transitive monodromy" is actually a holomorphic Lefschetz fibration. Auroux used this result to derive a stable classification of genus 2 Lefschetz fibrations (Section 3.4), and soon generalized it to $g \geq 3$ (Section 3.5), which also led to an analogous stable classification of symplectic 4-manifolds. Also, Siebert and Tian considered hyperelliptic Lefschetz fibrations, which have certain symmetry on Σ_g, and showed that they can be realized as branched covers over $\mathbb{S}^2$-fibrations (Section 3.5). In this case more tools in algebraic geometry are available, at the price that only a proper subset of all Lefschetz fibrations is considered.

Chapter 2

Symplectic 4-Manifolds

2.1 Basic Definitions

Definition 2.1. *Let V be a finite-dimensional vector space. A skew-symmetric bilinear map $\Omega : V \times V \to \mathbb{R}$ is said to be non-degenerate if for all $v \in V - \{0\}$, there exists $u \neq 0$ such that $\Omega(v, u) \neq 0$.*

Definition 2.2. *Let X be a smooth manifold and denote its tangent space at p by T_pX. A 2-form ω defined on X is said to be non-degenerate if $\omega_p : T_pX \times T_pX \to \mathbb{R}$, as a skew-symmetric bilinear map, is non-degenerate for all $p \in X$.*

Definition 2.3. *A symplectic form on a smooth manifold X is a smooth, closed and non-degenerate 2-form. A symplectic manifold is a pair (X, ω), where X is a smooth manifold and ω is a symplectic form.*

Definition 2.4. *Two symplectic manifolds (X_1, ω_1) and (X_2, ω_2) are said to be symplectomorphic if there exists a diffeomorphism $\phi : X_1 \to X_2$ such that $\phi^*\omega_2 = \omega_1$.*

It is immediate from definition that X is even-dimensional. One may also check that ω^n is a non-degenerate top form of X; thus every symplectic manifold is canonically oriented by ω^n.

There are also some classes of manifolds which are in close relations with symplectic manifolds:

Definition 2.5. *Let X be a $2n$-dimensional smooth manifold. An almost complex structure J on X is a smooth field of linear maps on the tangent spaces:*

$$p \mapsto J_p : T_pX \to T_pX, \quad J_p^2 = -\mathrm{Id}.$$

The pair (X, J) is called an almost complex manifold.

An almost complex structure naturally corresponds to a non-degenerate 2-form (but not necessarily closed); hence, an almost complex manifold is also even dimensional and canonically oriented.

Definition 2.6. *A fiber bundle consists of (X, B, π, F), where X, B, F are topological spaces and $\pi : X \to B$ is a continuous surjective map satisfying the local trivialization condition: for all $x \in X$, there exists a neighbourhood U of $\pi(x)$ and a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that $\pi|_{\pi^{-1}(U)} = \mathrm{proj} \circ \phi$, where* $\mathrm{proj} : U \times F \to U$ *is the projection to the first component.*

We are only interested in the case that X, B, F are smooth manifolds, π is smooth and that ϕ is a diffeomorphism.

2.2 Simple Examples of Symplectic Manifolds

We give a few easy examples first.

1. $\mathbb{R}^{2n}$ is a symplectic manifold for all $n \in \mathbb{Z}^+$, with the standard symplectic form being $\sum_{i=1}^{n} dx_i \wedge dy_i$.

2. Any compact oriented surface Σ, in particular $\mathbb{S}^2$, is symplectic, because any orientation form on Σ is closed, non-degenerate 2-form.

3. Let (X_1, ω_1) and (X_2, ω_2) be two symplectic manifolds, and $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$ be projection maps. Then $p_1^*(\omega_1) + p_2^*(\omega_2)$ is a symplectic form in $X_1 \times X_2$.

One naturally asks whether there are ways to construct more interesting examples of symplectic manifolds, or on the other side, whether there are any obstructions for a smooth manifold to be symplectic. The second question has a partial answer that can be answered by following propositions:

Proposition 2.1. *If (X, ω) is a closed symplectic manifold, then $\omega \neq 0 \in H^2_{\mathrm{dR}}(X)$.*

Proof. Assume that $H^2_{\mathrm{dR}}(X) = 0$. Then $\omega = d\tau$ where τ is a 1-form. By Stokes' theorem, we have

$$\int_X \omega^n = \int_X (d\tau)^n = \int_X d(\tau \wedge (d\tau)^{n-1}) = 0.$$

However, note that $\int_X \omega^n \neq 0$ for any non-degenerate smooth 2-form ω. Therefore we have a contradiction. □

From this we have more examples:

1. $\mathbb{S}^3 \times \mathbb{S}^1$ admits no symplectic structure as $H^2_{\mathrm{dR}}(\mathbb{S}^3 \times \mathbb{S}^1) = 0$.

2. By the same reason, $\mathbb{S}^{2n}$ admits no symplectic structure for any $n \geq 2$.

3. The compactness assumption of the proposition cannot be dropped. $\mathbb{R}^{2n}$ admits a symplectic structure even though $H^2_{\mathrm{dR}}(\mathbb{R}^{2n}) = 0$.

Proposition 2.2. *Any symplectic manifold (X, ω) admits a compatible almost complex structure J.*

Proof. First we prove this statement at vector space level. Let Ω be a non-degenerate skew-symmetric bilinear form on a vector space V, and let G be any inner product on V. Then there exists a skew-symmetric linear map A :

$V \to V$ such that $\Omega(u,v) = G(Au,v)$. From this one may check that AA^* is positive definite, and so diagonalizes with positive eigenvalues λ_i, i.e. $AA^* = B \cdot \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{2n}) \cdot B^{-1}$. We define

$$\sqrt{AA^*} := B \cdot \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_{2n}}) \cdot B^{-1},$$

and that

$$j = (\sqrt{AA^*})^{-1} A.$$

One can check that j is an almost complex structure on V. This proof actually carries to manifold level because the almost complex structures constructed by the above method vary smoothly with the symplectic structures. □

The almost complex structure J constructed in the proposition is also *ω-compatible*, meaning that $\omega(Ju, Jv) = \omega(u,v)$ for all $p \in X$ and $u, v \in T_pX$, and that $\omega(u, Ju) > 0$ for all $p \in X, u \in T_pX - \{0\}$.

Note that not every almost complex manifold is symplectic. $\mathbb{S}^3 \times \mathbb{S}^1$ admits a complex structure, in particular an almost complex structure, because it can be seen as the quotient of $\mathbb{C}^2 - \{0\}$ by the holomorphic map $(z_1, z_2) \mapsto (2z_1, 2z_2)$. However, as we have seen, $\mathbb{S}^3 \times \mathbb{S}^1$ cannot be symplectic. On the other hand, Gromov proved that every open manifold (that is, non-compact or has non-empty boundary) admits a symplectic structure if and only if it admits a non-degenerate form; as a consequence, every open almost complex manifold is symplectic (see [McS], Section 4.1).

2.3 A Theorem of Thurston

From now on, all manifolds are assumed to be compact, oriented and without boundary, unless otherwise stated.

We have seen that the product of two symplectic manifolds is symplectic. A product of symplectic manifolds can be seen as a trivial bundle over a symplectic manifold with symplectic fibers, so one naturally expects that the total space of a fiber bundle with analogous symplectic data is also symplectic. Indeed this is the case as shown by Thurston ([Th]):

Theorem 2.1. *Let $\pi : X^{2n} \to Y^{2n-2}$ be a fiber bundle with fiber F. Assume that X is connected, (Y, ω_Y) is symplectic and $[\pi^{-1}(y)] \neq 0 \in H_2(X, \mathbb{R})$ for all $y \in Y$. Then X admits a symplectic structure ω such that π has symplectic fibers, that is, $\omega|_F$ is symplectic for any fiber F.*

It is a consequence of Sard's theorem that $F \cong \Sigma_g$, where Σ_g denotes a compact oriented surface of genus g, and we have seen that Σ_g is symplectic. Therefore, the symplectic data of the fiber are already implied by the assumption that X is just two dimensions higher than Y. There is a generalization of the theorem to fiber bundles $\pi : X^{2n} \to Y^{2k}$ for any $n > k$, but then we need to assume, at least, that the fibers themselves are symplectic, and a statement free from symplectic assumptions on the fibers is not possible. On the other hand, if X is just 4-dimensional (take $n = 2$), which we are more interested in, then all symplectic assumptions of the theorem automatically hold, which implies that some symplectic 4-manifolds can be constructed by purely topological data.

With this theorem, one can generate symplectic manifolds by "twisting" product spaces. For example, let $\mathbb{T}^2$ be the 2-dimensional torus and $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ be a *Dehn twist*. Defining $M = \mathbb{T}^2 \times I/((x,0) \sim (\phi(x),1))$ and taking $\mathbb{S}^1 \cong [0,1]/(0 \sim 1)$, we have an $\mathbb{T}^2$-bundle over $\mathbb{S}^1$ given by the obvious projection $M \to \mathbb{S}^1$. Crossing this bundle with $\mathbb{S}^1$ yields $\pi : M \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, a $\mathbb{T}^2$-bundle over $\mathbb{T}^2$. It is clear that $[\pi^{-1}(y)] \neq 0$. It is easy to show that $M \times \mathbb{S}^1$ cannot be written as a product of symplectic manifolds.

However, not all total spaces of torus bundles are symplectic. The *Hopf fibration* $\mathbb{S}^3 \to \mathbb{S}^2$ is an $\mathbb{S}^1$-bundle over $\mathbb{S}^2$, and crossing $\mathbb{S}^3$ with $\mathbb{S}^1$ yields a $\mathbb{T}^2$-bundle over $\mathbb{S}^2$. However, as we have seen before, $\mathbb{S}^1 \times \mathbb{S}^3$ does not admit a symplectic structure.

Proof of Theorem 2.1. :

Step 1. As $[\pi^{-1}(y)] \neq 0$, there exists a closed 2-form α on X such that $\int_F \alpha \neq 0$. We claim that there exists a 2-form β on X, not necessarily closed, which is non-degenerate on each fiber, and that $\int_F \beta = \int_F \alpha$. To find such β, first let $(\rho_\gamma, U_\gamma)_{\gamma\in\Gamma}$ be a partition of unity on Y such that π can be trivialized over any U_γ. Pull back $\{\rho_\gamma\}_{\gamma\in\Gamma}$ to X to obtain a partition of unity $(\tilde{\rho}_\gamma, \pi^{-1}(U_\gamma))$ on X. Now every fiber is canonically oriented by X, so for any $\gamma \in \Gamma$, we may pick a fiber F_γ in $\pi^{-1}(U_\gamma)$ and an area form β_γ (or negative of it) on F_γ such that $\int_{F_\gamma} \beta_\gamma = \int_{F_\gamma} \alpha$. Note that we have used the assumption $\int_F \alpha \neq 0$. Let $h_\gamma : \pi^{-1}(U_\gamma) \to U_\gamma \times F_\gamma$ be local trivalizations of π, and $p_\gamma : U_\gamma \times F_\gamma \to F_\gamma$ be the projection. Then $\tilde{\beta}_\gamma := (h_\gamma \circ p_\gamma)^*(\beta_\gamma)$ is defined on the whole $\pi^{-1}(U_\gamma)$, non-degenerate on each fiber over U_γ and $\int_{F_\gamma} \tilde{\beta}_\gamma = \int_{F_\gamma} \beta_\gamma$. Now define $\beta = \Sigma\tilde{\rho}_\gamma\tilde{\beta}_\gamma$. It is easy to check that we have $\int_F \beta = \int_F \alpha$ on any fiber F. Since all $\tilde{\beta}_\gamma$ defined on a fiber F are non-degenerate and induce the same orientation on F, then so does their convex combination. Thus β is the form we want to have.

Step 2. Since $\int_F \beta = \int_F \alpha$ for any fiber F, there exists ξ_F on each fiber F such that $\beta|_F - \alpha|_F = d(\xi_F)$. This equation can be solved in a small neighbourhood of the base point of F, thus there exists a finite cover $\{V_\delta\}_{\delta\in\Delta}$ of Y subordinate to $\{U_\gamma\}_{\gamma\in\Gamma}$, which pulls back to $\{\pi^{-1}(V_\delta)\}_{\delta\in\Delta}$, such that for each δ, there exists a 1-form ξ_δ defined on $\pi^{-1}(V_\delta)$ such that $(\beta - \alpha)|_{\pi^{-1}(V_\delta)} = d(\xi_\delta)$. Now we splice the ξ_δ*'s* together. Again, let $\{\sigma_\delta\}_{\delta\in\Delta}$ be partition functions for $\{V_\delta\}_{\delta\in\Delta}$ and $\{\tilde{\sigma}_\delta\}_{\delta\in\Delta}$ be their pull backs. Define $\xi = \Sigma\tilde{\sigma}_\delta\xi_\delta$. ξ is a globally defined 1-form, and by some calculation, one can show that $\beta|_F - \alpha|_F = (d\xi)|_F$. It follows that $\tilde{\alpha} := \alpha + d\xi$ is a closed form on X and is non-degenerate on each fiber.

Step 3. Let ω_Y be the symplectic form on Y. Then $\pi^*(\omega_Y)$ is not a symplectic form; in fact, for any $p \in X$ with $\pi(p) = q$, $(\pi^{-1}(\omega_Y))_p = 0$ on $T_p(\pi^{-1}(q))$ and is non-degenerate in the other directions. However, with the $\tilde{\alpha}$ from Step 2, it follows from compactness argument that $\omega := \pi^*(\omega_Y) + t\tilde{\alpha}$ is a symplectic form on X for any sufficiently small t. $\omega|_F = t\tilde{\alpha}$ which is symplectic on F by construction. □

A natural way to generalize fiber bundle structure is to allow π to have critical points. With some control to the behaviour around the critical points, we obtain the definition of Lefschetz fibration as follows.

Definition 2.7. *Let X be a compact, oriented, smooth 4-manifold and Σ be a compact, oriented surface, both possibly with boundary. A Lefschetz fibration on X over Σ is a smooth, onto map $\pi : X \to \Sigma$ such that*

1. $\pi^{-1}(\partial\Sigma) = \partial X$, *and*
2. *each critical point of π lies in the interior of X and has orientation-preserving charts around p and $\pi(p)$ such that $\pi(z_1, z_2) = z_1^2 + z_2^2$.*

$\pi : X \to \Sigma$ is called a genus g Lefschetz fibration if its generic fiber is Σ_g, the compact oriented surface of genus g.

In particular, all the critical points of π are non-degenerate. Therefore, one may take Lefschetz fibrations as the complex analogue of Morse function.

Let C denote the set of critical points of π. Then $\pi|_{X-\pi^{-1}(\pi(C))}$ is just a fiber bundle. Therefore, a Lefschetz fibration is, in some sense, almost a fiber bundle, but has additional complex data around the critical points. One may assume that π is injective on C by perturbing π. The preimage of π of a critical value is called a *singular fiber*, and that of a regular value is called a *generic fiber*.

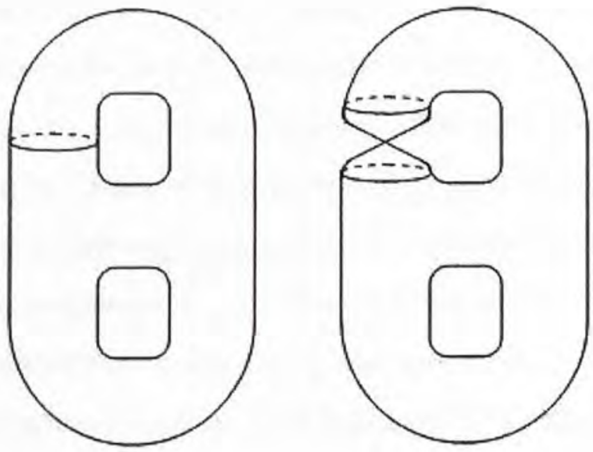

Figure 2.1: A generic fiber and a singular fiber.

Just like a fiber bundle, all the generic fibers are compact oriented surfaces and are diffeomorphic. If the fiber is connected, it has to be Σ_g. Choose a chart around a critical point p as in (2) of the above definition. Then $\pi^{-1}(0)$ can be locally described as $\{(z_1, z_2)|z_1 = 0 \text{ or } z_2 = 0\}$, which is a pair of intersecting planes (a nodal singularity). Therefore a singular fiber is a smoothly immersed surface. A singular fiber is obviously not diffeomorphic to a generic fiber but is still homologous to it. It was shown by Gompf ([GS], Prop. 10.2.18) that a Lefschetz fibration satisfying the same homological condition as Theorem 2.1 can also be shown to admit symplectic structure:

Theorem 2.2. *Let X be a compact, oriented 4-manifold without boundary, and assume that X admits a Lefschetz fibration $\pi : X \to \Sigma$. Let $[F]$ denote the homology class of a fiber F. Then X admits a symplectic structure with symplectic fibers if and only if $[F] \neq 0$ in $H_2(X, \mathbb{R})$.*

Sketch of proof. The proof of this theorem is similar to Theorem 2.1. However, there is a crucial point in the proof which utilizes the behaviour of the critical points. The rough idea is as follows. One can choose a chart U_p around a critical point p such that $\pi(z_1, z_2) = z_1^2 + z_2^2$ prescribed by the definition of Lefschetz fibration. Then one considers the symplectic form $\omega_{U_p} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on U_p (where $z_i = x_i + y_i$), which is compatible with the complex structure. Let $F_p = \pi^{-1}(\pi(p))$. Since π is a holomorphic map in U_p, $F_p \cap U_p - \{p\}$ is a holomorphic curve and $\omega_{U_p}|_{F_p \cap U_p - \{p\}}$ is a symplectic form, in particular non-degenerate. This symplectic form can be extended to the whole $F_p - \{p\}$; thus we get two symplectic forms, one on U_p and one on $F_p - \{p\}$, agreeing at the intersection. A neighbourhood of F_p is not a product; instead, it is a product with a 2-handle attached to it (see the next section). Nonetheless, it is possible to extend the forms on U_p and F_p to the neighbourhood of F_p and still satisfying some local exactness condition as in Step 2 of the proof of Theorem 2.1; see [GS] for details. The rest of the proof consists of splicing forms and compactness argument, which is the same as that of Theorem 2.1. □

A few remarks on this theorem:

1. The assumption that the local charts around the critical points of π are orientation-preserving turns out to be crucial. If this assumption is dropped, we get a wider class of fibrations called *achiral Lefschetz fibrations*. Without this assumption, X can no longer be shown to possess symplectic structure. In fact, there are easy examples of achiral Lefschetz fibrations in which the total spaces do not admit symplectic structures; again, see [GS], Section 8.4.

2. It follows that *Lefschetz pencils* (see next section) also admit symplectic structures by considering blow-ups and blow-downs.

2.4 Lefschetz Pencils

We first start with the construction of a Lefschetz pencil on an algebraic surface $X \subset \mathbb{CP}^N$. Let A be a generic linear subspace of complex codimension 2, such that

$A = \{z \in \mathbb{CP}^N : p_0(z) = p_1(z) = 0\}$ for some homogeneous linear polynomials p_0 and p_1. A is generic in the sense that A intersects X transversely at a finite set of points B called the base locus. The set of all hyperplanes passing through A can be parametrized by $\mathbb{CP}^1$; each such plane can be represented by $P_y := \{y_0p_0(z) + y_1p_1(z) = 0\}$, where $y = [y_0, y_1] \in \mathbb{CP}^1$. These planes intersect X in a family of complex curves $\{F_y \mid y \in \mathbb{CP}^1\}$; note that finitely many F_y's are possibly singular. Since $\bigcup P_y = \mathbb{CP}^N$ and $P_y \cap P_{y'} = A$ for any distinct $y, y' \in \mathbb{CP}^1$, we see that $\bigcup F_y = X$ and $F_y \cap F_{y'} = B$ for distinct $y, y' \in \mathbb{CP}^1$. So the canonical map $\mathbb{CP}^N - A \to \mathbb{CP}^1$ restricts to a holomorphic map

$$\pi : X - B \to \mathbb{CP}^1,$$

determined by $\pi^{-1}(y) = F_y$. The set of base locus B and the map $\pi : X - B \to \mathbb{CP}^1$ is called a *Lefschetz pencil* on an algebraic surface X.

π has two important features. Firstly, since A intersects X transversely, each F_y is smooth near B and π can be identified as the projectivization $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$ near B. Secondly, as A is chosen to be generic, π will also be generic, meaning that all critical points of π are complex quadratic. In fact, these are the ingredients for defining Lefschetz pencils in the smooth category:

Definition 2.8. *Let X be a closed 4-manifold without boundary. A Lefschetz pencil on X is a non-empty finite set $B \subset X$, called the base locus, and a map $\pi : X - B \to \mathbb{CP}^1$ satisfying the following:*

1. *For any $b \in B$, there exists an orientation-preserving chart (U, h), $h : (U, b) \to (\mathbb{C}^2, 0)$, such that $(\pi \circ h^{-1})|_{\mathbb{C}^2-\{0\}}$ is the projectivization $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$.*

2. *Each critical point p of π has an orientation-preserving chart (V, k) around p such that $\pi \circ k^{-1} = z_1^2 + z_2^2$.*

The local condition for the critical points of π for Lefschetz fibrations and Lefschetz pencils is the same. So the only difference is that π is not defined over all X for a Lefschetz pencil; in fact, π cannot be smoothly extended to the whole X as the projectivization $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$ cannot be smoothly extended to the origin. However, a Lefschetz pencil can indeed be "desingularized" by *blowing up* at the base locus, and the result will be a Lefschetz fibration. Intuitively, one replaces a point in B by a copy of $\overline{\mathbb{CP}^2}$ ($\mathbb{CP}^2$ with reversed orientation), so that π can be extended naturally to a Lefschetz fibration on $X\#\overline{\mathbb{CP}^2}$ without producing any critical points.

Now we are ready to state the result by Donaldson ([D1]) regarding the existence of Lefschetz pencils on symplectic 4-manifolds:

Theorem 2.3. *Let (X, ω) be a closed symplectic 4-manifold and suppose $\omega \in H^2_{\mathrm{dR}}(X;\mathbb{Z})$ is an integral form. Then for a sufficiently large k, there exists a Lefschetz pencil on X with symplectic fibers which are homologous to the Poincare dual of $k[\omega]$.*

The requirement that ω is integral is only for convenience; it is easy to show that any closed symplectic manifold (X, ω) admits another symplectic form ω' which has integral cohomology class.

The proof of the theorem is a generalization of techniques in complex geometry. In complex geometry, one can consider a holomorphic line bundles L over a complex manifold X and take high tensor powers of L so that it contains desired holomorphic sections. Now every symplectic manifold has a compatible almost complex structure, so one can consider J-holomorphic line bundles. However, the existence of J-holomorphic sections in L^k is not guaranteed; Donaldson turned to prove, by analysis on L^k, the existence of "approximately holomorphic sections" which also served his purpose of showing the existence of a (real) codimension-2 symplectic submanifold of certain homology class ([D2]). Then similar techniques are used to show the existence of Lefschetz pencils.

To conclude, closed symplectic 4-manifolds, Lefschetz pencils and Lefschetz fibrations can be said to be equivalent, as all symplectic 4-manifolds admit Lefschetz pencils, Lefschetz pencils can be blown up to yield Lefschetz fibrations, and both Lefschetz pencils and Lefschetz fibrations admit symplectic structures (except certain torus bundles). This motivates us to study symplectic geometry through Lefschetz fibrations.

Chapter 3

Classification of Lefschetz Fibrations

3.1 Definitions

We restate the definition of Lefschetz fibration.

Definition 3.1. *Let X be a compact, oriented, smooth 4-manifold and Σ be a compact, oriented surface, both possibly with boundary. A Lefschetz fibration on X over Σ is a smooth, onto map $\pi : X \to \Sigma$ such that*

a. $\pi^{-1}(\partial\Sigma) = \partial X$, *and*

b. each critical point of π lies in Int X and has an orientation-preserving charts around p and $\pi(p)$ such that $\pi(z_1, z_2) = z_1^2 + z_2^2$ in the chart.

$\pi : X \to \Sigma$ is called a genus g Lefschetz fibration if a generic fiber of it is Σ_g.

Notation 3.1. *A Lefschetz fibration is simply denoted by its map π when no ambiguity arises.*

Definition 3.2. *Two Lefschetz fibrations $\pi_1 : X_1 \to \Sigma_1$ and $\pi_2 : X_2 \to \Sigma_2$ are said to be isomorphic if there exist orientation-preserving diffeomorphisms*

$\psi : X_1 \to X_2$, $\phi : \Sigma_1 \to \Sigma_2$ such that $\phi \circ \pi_1 = \pi_2 \circ \psi$. Such an isomorphism is denoted by (ψ, ϕ).

It was remarked in the previous chapter that all Lefschetz fibrations are assumed to be injective on critical sets. Here we introduce another useful assumption on Lefscehtz fibrations:

Definition 3.3. *A Lefschetz fibration is relatively minimal if no fibers contain a sphere of self-intersection -1.*

If a Lefschetz fibration contains a sphere of self-intersection -1 in a fiber (which must be singular), then it can be removed by a blow down while still preserving the fibration. Therefore, a Lefschetz fibration is relatively minimal if and only if it cannot be seen as a fiber-preserving blow up of another Lefschetz fibration. We will also see that non-minimal Lefschetz fibration has critical points with null-homotopic *vanishing cycles*, which is unwanted in subsequent work. Therefore, we also assume that the Lefschetz fibrations discussed are relatively minimal unless otherwise stated.

The case of interest is $\Sigma = \mathbb{S}^2$ as it contains all the Lefschetz fibrations that arise from Lefschetz pencils. We also consider the case $\Sigma = \mathbb{D}^2$ as a tool for studying the case $\Sigma = \mathbb{S}^2$. Given a Lefschetz fibration over $\mathbb{S}^2$, we may split $\mathbb{S}^2$ into $\mathbb{D}^2 \cup \mathbb{D}^2$ so that all of the critical values are contained in the interior of one of the disks. The restriction of π to the disk containing the critical values gives a Lefschetz fibration over $\mathbb{D}^2$ which is trivial over $\mathbb{S}^1 = \partial\mathbb{D}^2$). Conversely, given a Lefschetz fibration over $\mathbb{D}^2$, $\mathbb{S}^1 = \partial\mathbb{D}^2$ contains no critical values, and so $\pi^{-1}(\mathbb{S}^1)$ fibers over $\mathbb{S}^1$. If π is indeed the trivial bundle over $\mathbb{S}^1$, then π can be extended to give a Lefschetz fibration over $\mathbb{S}^2$.

3.2 Handlebody Decomposition

In this section, the classical work of Kas ([K]) is sketched so as to introduce one of the fundamental point of views of the study of Lefschetz fibrations. By a Morse-theoretic argument, a genus g Lefschetz fibration $\pi : X \to \Sigma$ is seen to be constructed by attaching 2-handles to $\mathbb{D}^2 \times \Sigma_g$ with some special framings due to the existence of certain Morse function on X, and then closing up the boundary of the resulting manifold. Moreover, the framings can be determined by *vanishing cycles* and eventually allows a genus g Lefschetz fibration to be described combinatorially. Kas' work was originally on more general Lefschetz fibrations over $\mathbb{S}^2$, which is a map $\pi : X \to \mathbb{S}^2$ where X can be of any even dimension greater than 2 and has analogous local conditions for the critical points. In the particular case $\dim X = 4$, some of the results can be simplified.

We first recall some facts about handlebody decomposition and Morse theory (cf. [GS], [K]). Let N be a manifold with boundary and let $n = \dim N$. Let $\Phi : \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \to \partial N$ be a smooth imbedding.

Definition 3.4. *The manifold $N \cup_\Phi (\mathbb{D}^k \times \mathbb{D}^{n-k})$, obtained by identifying each point of $\mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \subset \partial(\mathbb{D}^k \times \mathbb{D}^{n-k})$ with its image under Φ and then straightening corners, is called the manifold obtained from N by attaching a k-handle along Φ.*

Definition 3.5. *A handle decomposition of a manifold N is a sequence of submanifolds $N_0 \subset N_1 \subset N_2 \ldots N_k = N$, where N_0 is a disjoint union of n-balls, and each N_j is a submanifold obtained by attaching j-handles to N_{j-1}.*

We turn our attention to the attaching map Φ. The diffeomorphism type of $N \cup_\Phi (\mathbb{D}^k \times \mathbb{D}^{n-k})$ only depends on the smooth isotopy class of Φ. Now denote Φ_0 by the restriction of Φ to $\mathbb{S}^{k-1} \times \{0\}$. It follows from tubular neighborhood theorem that Φ is determined, up to smooth isotopy, by a bundle isomorphism:

$$\Phi' : \varepsilon^{n-k} \to \nu$$

where ε^{n-k} is the trivial $\mathbb{D}^{(n-k)}$-bundle on $\mathbb{S}^{k-1}$, and ν is the normal bundle of $\mathbb{S}^{k-1}$ in ∂N under the imbedding Φ_0. The isotopy class of Φ' is called the *framing* of Φ_0. Therefore, the isotopy class of Φ is determined by a smooth isotopy class of imbeddings $\Phi_0 : \mathbb{S}^{k-1} \to \partial N$; and a framing of Φ_0. In general, given an imbedding $\Phi_0 : \mathbb{S}^{k-1} \to \partial N$, the framings are classified by $\pi_{k-1}(SO(n-k))$.

Now let $\pi : X \to \mathbb{S}^2$ be a genus g Lefschetz fibration with critical points $C = \{c_1, c_2, \ldots c_\mu\}$. We may identify $\mathbb{S}^2$ with $\mathbb{C} \cup \infty$ such that ∞ is a regular value of π, and for convenience, we perturb π so that $\pi(c_k) = \exp \frac{2\pi i k}{\mu}$. It is then easy to check that $f : X - \pi^{-1}(\infty) \to \mathbb{R}$, defined by $f(x) = |\pi(x)|^2$ is a Morse function. Further, the critical points of f and π are the same, 1 is the only critical value of f, and all of the critical points are of index 2. Define $X_y = \{z \in X : f(z) \leq y\}$. Then for any $\epsilon > 0$, $X_{1-\epsilon} \cong \Sigma_g \times \mathbb{D}^2$, and $X_{1+\epsilon}$ is diffeomorphic to the manifold obtained from $\Sigma_g \times \mathbb{D}^2$ by attaching μ 2-handles by the imbeddings $\Phi_j : \mathbb{S}^1 \times \mathbb{D}^2 \to \partial(\Sigma_g \times \mathbb{D}^2) = \Sigma_g \times \mathbb{S}^1, j = 1, \ldots \mu$, so that the Φ_j's have disjoint images.

So we are interested in the case where $n = 4$ and $k = 2$, and the group of framings given a Φ_0 is $\pi_1(SO(2)) \cong \mathbb{Z}$. However, in our situation, the framing is not completely arbitrary. As shown by Kas, the existence of the Morse function f causes the group of framings $\pi_{k-1}(SO(n-k))$ to be reduced to another but smaller group of framings $\pi_{k-1}(SO(n-k-1))$. When $n = 4$ and $k = 2$, the group is reduced from $\mathbb{Z}$ to trivial group; thus, given the imbedding Φ_0, the framing is essentially unique.

We will state the following lemma and theorem in the dimension of our interest, although the proofs are no different for general dimensions.

Lemma 3.1. *The Morse function f can be perturbed and Φ_j can be chosen such* *that* $\Phi_j(\mathbb{S}^1 \times 0) \subset \pi^{-1}(z_j)$ *for some* $z_j \in \mathbb{S}^2$.

By Lemma 3.1, we may identify $\pi^{-1}(X_{1-\epsilon}) \cong \Sigma_g \times \mathbb{D}^2$ and define $\phi_j : \mathbb{S}^1 \to \Sigma_g$ to be the imbedding Φ_j restricted to $\mathbb{S}^1 \times \{0\}$.

Theorem 3.1. *Fix the following notations:*

- ν_1 *denotes the normal bundle of* $\phi_j(\mathbb{S}^1)$ *in* Σ_g.

- ν *denotes the normal bundle of* $\phi_j(\mathbb{S}^1)$ *in* $f^{-1}(1-\epsilon)$.

- τ *denotes the tangent bundle of* $\mathbb{S}^1$.

- ε^2 *and* ε *denote the trivial* $\mathbb{D}^2$*-bundle and* $\mathbb{D}^1$*-bundle over* $\mathbb{S}^1$ *respectively.*

There exists a natural bundle isomorphism k

$$\varepsilon^2 \xrightarrow{k} \tau \oplus \varepsilon,$$

such that for each framing $\Phi_j' : \varepsilon^2 \to \nu$ *of* Φ_j, *there exists a bundle isomorphism* $\phi_j' : \tau \to \nu_1$ *such that* ${\Phi_j}'$ *can be written as*

$$\varepsilon^2 \xrightarrow{k} \tau \oplus \varepsilon \xrightarrow{\phi_j' \oplus 1} \nu_1 \oplus \varepsilon \xrightarrow{\sim} \nu.$$

Therefore, the group of the framing $\varepsilon^2 \to \nu$ is reduced to $\tau \to \nu_1$. However, there is only one isomorphism class of bundle isomorphisms from τ to ν_1, so the whole map Φ_j is determined by $\phi_j(\mathbb{S}^1)$ up to isotopy.

Definition 3.6. $\phi_j(\mathbb{S}^1) = \gamma_j \subset \Sigma_g$ *is called the vanishing cycle for the Lefschetz fibration* $\pi : M \to \mathbb{S}^2$ *associated to the critical value* $\pi(c_j)$.

Although ϕ_j' is the trivial bundle map, the whole framing Φ_j is not so. There is a natural way to trivialize the normal bundle of γ_j in $\Sigma \times \mathbb{S}^1$, since the normal bundle of γ_j in Σ_g is trivial, and the normal bundle of Σ_g in $\Sigma_g \times \mathbb{S}^1$ is trivial. This gives the "natural framing". One can see that the framing $\Phi_j' : \varepsilon^2 \to \nu$

given by the theorem has framing -1 relative to the natural framing. That is, if we identify ν with ε^2 by the "'natural framing"', then $\Phi'_j : \varepsilon^2 \to \nu$ is given by $\mathbb{S}^1 \to SO(2) \cong \mathbb{S}^1$ of degree -1. Intuitively, this captures the self-intersection of the singular fiber.

The vanishing cycle γ completely determines the topology of a neighbourhood of a singular fiber since it completely determines the framing. The diffeomorphism types of a neighbourhood of a singular fiber are also quite limited. For any two non-separating smooth $\mathbb{S}^1$ embedded in Σ_g, there exists a diffeomorphism of Σ_g which brings one to the other, and the handlebody decomposition associated to the two $\mathbb{S}^1$ are the same. Hence, for a fixed g, there are $\left[\frac{g}{2}\right] + 1$ diffeomorphism types of a neighbourhoods of a singular fiber, one given by any non-separating vanishing cycles, the rest given by separating cycles in which the portion with smaller genus has genus 1 to $\left[\frac{g}{2}\right]$ (nullhomotopic vanishing cycles are not considered as such Lefschetz fibrations are not relatively minimal). For example, when $g = 1$, there is only one diffeomorphism type of neighbourhood of a singular fiber.

Since the diffeomorphism types of a neighbourhood of a singular fiber are quite limited, the variation of Lefschetz fibration mainly comes from how those neighbourhoods are glued together. To study it we have to describe the vanishing cycles associated to different critical points with a *common identification* of Σ_g but not separately, so that the vanishing cycles of different critical points can be compared. A more precise description is as follows. Consider a genus g Lefschetz fibration over $\mathbb{D}^2$ or $\mathbb{S}^2$, with critical values $A = \{a_1, a_2, \ldots, a_\mu\}$, and let $a \in \mathbb{D}^2$ be a regular value (for Lefschetz fibrations over $\mathbb{D}^2$, assume that $A \cup \{a\}$ lies inside a large disk D). By some perturbation, we can assume that there are disjoint arcs s_i joining a and a_i, disjoint positively oriented small circles Z_i around a_i, and a small positively oriented circle around a that cuts the arcs in

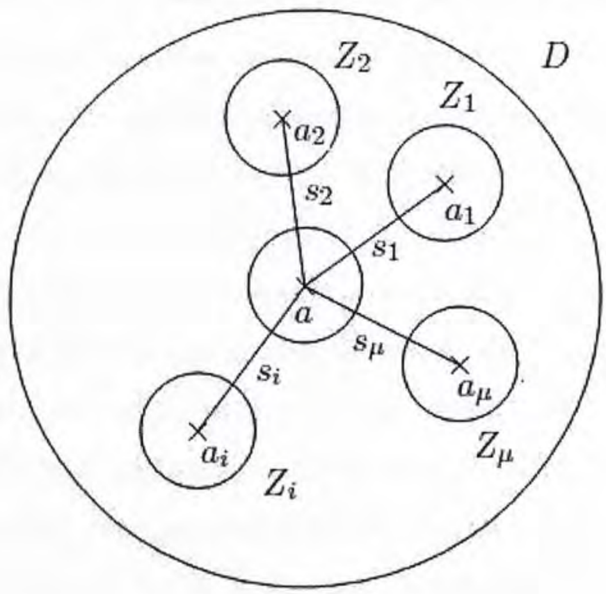

Figure 3.1: Common identification and ordering critical points.

the order $s_1, s_2, \ldots, s_\mu$ (see Figure 3.1), hence inducing an order of critical values $a_1, a_2, \ldots, a_\mu$. $\pi^{-1}(a) = \Sigma_g$ serves as the common identification for describing the vanishing cycles γ_i associated to $\pi^{-1}(a_i)$ by transferring every vanishing cycle to $\pi^{-1}(a)$ through the arcs s_i. Hence we obtain a μ-tuple of vanishing cycles $(\gamma_1, \gamma_2, \ldots, \gamma_\mu)$ by Theorem 3.1. The μ-tuple is well-defined if the choices of the common identification and the order of the critical values are fixed. Different choices may yield distinct tuples for the same Lefschetz fibration; this will be discussed in detail in the next section. With the same common identification, one define the *monodromy representation* of π, which is a map

$$\xi : \pi_1(\mathbb{S}^2 - A) \to \mathrm{MCG}(\Sigma_g),$$

where $\mathrm{MCG}(\Sigma_g) := \mathrm{Diff}_+(\Sigma_g)/\mathrm{Isot}(\Sigma_g)$ is the group of isotopy classes of orientation preserving diffeomorphisms of Σ_g, also called the mapping class group of Σ_g. Intuitively, for any loop Z in Σ which is disjoint from A, one notes that

$\pi|_{\pi^{-1}(Z)} : \pi^{-1}(Z_i) \to Z$ is a bundle over $\mathbb{S}^1$, but is possibly twisted as Z_i bounds a critical value of π; the map ξ keeps track of the twists and, as we will see, lead to a characterization of a Lefschetz fibration. More precisely, let $Z : I \to \Sigma - A$ be a loop in $\Sigma - A$, so that $Z(0) = Z(1)$. The pullback bundle $\pi' : Z^*(X) \to I$ is canonically trivial, so by identifying $Z^*(X)$ with $I \times \Sigma_g$, a diffeomorphism $k : (\pi')^{-1}(0) \to (\pi')^{-1}(1)$, up to isotopy, is induced. Then $(\pi')^{-1}(0)$ and $(\pi')^{-1}(1)$ are identified with $\pi^{-1}(a) = \Sigma_g$ through the arc s_i, obtaining a diffeomorphism $\tilde{k} : \Sigma_g \to \Sigma_g$, again defined up to isotopy. We define $\xi[Z] = [\tilde{k}]$, and it is easy to check that ξ is a homomorphism and well-defined with a fixed choice of common identification and order of critical values. For convenience, we call also the μ-tuple $(\xi([Z_1]), \xi([Z_2]), \ldots, \xi([Z_\mu]))$ the monodromy representation of π, and for each i, $(\xi[Z_i])$ the monodromy of π associated to a_i.

It turns out that the monodromy representation can be explicitly described in terms of the vanishing cycles. We first define *right-handed Dehn twist*.

Definition 3.7. *Let $\gamma = \phi(\mathbb{S}^1)$ be an embedded circle in Σ_g and $V \cong \mathbb{S}^1 \times [0,1]$ be a closed tubular neighbourhood of γ, where $\mathbb{S}^1 \times [0,1]$ has the product orientation. Define a diffeomorphism $\alpha : V \to V$ by $\alpha(\theta, t) = (\theta + 2\pi t, t)$. The right-handed Dehn twist with center γ is the diffeomorphism δ_ϕ of Σ_g, defined up to isotopy, given by*

$$\delta_\gamma(x) = \begin{cases} \alpha(x) & \text{if } x \in V \\ \mathrm{Id}(x) & \text{if } x \in \Sigma_g - \mathrm{Int}(V) \end{cases}$$

after smoothing at the boundary of V.

It is easy to see that the isotopy class of δ_γ only depends on that of γ. One may also define left-handed Dehn twist by changing α to $\alpha(\theta, t) = (\theta - 2\pi t, t)$. However, for the rest of this article, all Dehn twists concerned are assumed to be right-handed; therefore we simply call them *(positive) Dehn twists*.

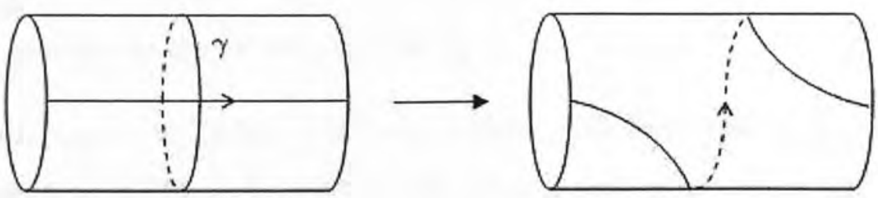

Figure 3.2: Right-handed Dehn twist about γ.

Theorem 3.2. *Let p be a critical value of π and γ be the vanishing cycle associated with p. Then the monodromy of π associated to p is isotopic to the right-handed Dehn twist δ_γ.*

Sketch of proof: Without loss of generality, assume $p = 1$. Let ϕ' denote the unique framing determined by ϕ with image U, $\iota : I \to \mathbb{S}^1$ be an embedding such that $p \in \iota(I)$ and let Z be a small circle around p. It is then possible to write down $\pi^{-1}(Z)$ explicitly as follows. Consider $\phi' \times \iota : U \times I \cong \mathbb{S}^1 \times \mathbb{D}^2 \to \Sigma_g \times \mathbb{S}^1$. Then

$$\pi^{-1}(Z) \cong (\Sigma_g \times I - U \times I)/\sim,$$

where $\sim$ is the equivalence relation on $\partial(U \times I)$ given by $(\xi, \eta) \mapsto (\eta, \xi)$. The resulting manifold can be given a smooth structure, and the monodromy can be calculated by constructing a flow on $\pi^{-1}(Z)$. □

It is immediate from the previous theorem that $\pi^{-1}(\partial D) = (\Sigma_g \times I)/((\delta_{\gamma_\mu} \circ \delta_{\gamma_{\mu-1}} \circ \ldots \circ \delta_{\gamma_1}(x), 0) \sim (x, 1))$. The composition $\delta_{\gamma_\mu} \circ \delta_{\gamma_{\mu-1}} \circ \ldots \circ \delta_{\gamma_1}$ of a Lefschetz fibration over $\mathbb{D}^2$ is quite arbitrary. For any $h \in \mathrm{MCG}(\Sigma_g)$, one can write it as the composition of right-handed Dehn twists. Then one can start with the trivial bundle $\Sigma_g \times \mathbb{D}^2 \to \mathbb{D}^2$ and attach 2-handles along the corresponding vanishing cycles with framing prescribed by Theorem 3.1. However, the situation for Lefschetz fibration over $\mathbb{S}^2$ is quite different:

Fact 3.1. *The composition $\delta_{\gamma_\mu} \circ \delta_{\gamma_{\mu-1}} \circ \ldots \circ \delta_{\gamma_1}$ of a Lefschetz fibration π over $\mathbb{S}^2$ must be isotopic to the identity map of Σ_g.*

Proof. $\pi|_{\pi^{-1}(\partial D)} : \pi^{-1}(\partial D) \to \partial D$ is a bundle with total space $\Sigma_g \times I/((\delta_{\gamma_\mu} \circ \delta_{\gamma_{\mu-1}} \circ \ldots \circ \delta_{\gamma_1}(x), 0) \sim (x, 1))$ and the natural projection. However, we may also see $\pi|_{\pi^{-1}(\partial D)}$ as $\pi|_{\pi^{-1}(\overline{\mathbb{S}^2 - \partial D})}$, which shows that it is a trivial bundle (as all critical points of π lie in $\mathrm{Int}(D)$). A classical result of classification of bundles over spheres asserts that two Σ_g-bundles over $\mathbb{S}^1$, which must be of the form $\Sigma_g \times I/((\phi_1(x), 0) \sim (x, 1))$ and $\Sigma_g \times I/((\phi_2(x), 0) \sim (x, 1))$, are isomorphic if and only if ϕ_1 and ϕ_2 belong to the same conjugacy class of $\mathrm{MCG}(\Sigma_g)$ (See [Ste], P.99). Hence, $\delta_{\gamma_\mu} \circ \delta_{\gamma_{\mu-1}} \circ \ldots \circ \delta_{\gamma_1}$ and the identity map are conjugate to each other. The result follows. □

This fact suggests to the following terminologies:

Definition 3.8. *A μ-tuple of vanishing cycles $(\gamma_1, \gamma_2, \ldots, \gamma_\mu)$ is said to be admissible if $\delta_{\gamma_\mu} \circ \delta_{\gamma_{\mu-1}} \circ \ldots \circ \delta_{\gamma_1}$ is isotopic to identity map. The composition $\delta_{\gamma_\mu} \circ \delta_{\gamma_{\mu-1}} \circ \ldots \circ \delta_{\gamma_1}$ is called the factorization of π.*

The final step is to close up the manifold $X_{1+\epsilon}$. In other words, we attach a copy of $\mathbb{D}^2 \times \Sigma_g$ to $X_{1+\epsilon}$, the attaching map $\tilde{\alpha} : \mathbb{S}^1 \times \Sigma_g \to \mathbb{S}^1 \times \Sigma_g$ being a fiber-preserving diffeomorphism, in the form $\tilde{\alpha}(\theta, x) = (\theta, \alpha_\theta(x))$, where $\alpha_\theta : \Sigma_g \to \Sigma_g$ is a diffeomorphism depending on θ. By changing the trivialization of $\mathbb{D}^2 \times \Sigma_g$, we may assume that $\alpha_0 = \mathrm{Id} \in \mathrm{Diff}(\Sigma_g)$, so that $\alpha_\theta \in \mathrm{Diff}_0(\Sigma_g)$ for any θ. Therefore, an attaching map $\tilde{\alpha}$ corresponds naturally to a closed loop α, given by $\theta \mapsto \alpha_\theta$, in $\mathrm{Diff}_0(\Sigma_g)$, the diffeomorphism group of Σ_g. Conversely, a closed loop $\theta \mapsto \alpha_\theta$ in $\mathrm{Diff}_0(\Sigma_g)$ corresponds to the attaching map $\tilde{\alpha}(\theta, x) = (\theta, \alpha_\theta(x))$.

If α and α' are isotopic loops in $\mathrm{Diff}_0(\Sigma_g)$, then the corresponding attaching maps will be isotopic. Thus the diffeomorphic type of the resulting manifold only depends on the homotopy class of α; in other words, the diffeomorphic types are classified by $\pi_1(\mathrm{Diff}_0(\Sigma_g))$. For $g \geq 2$, it has been known that $\pi_1(\mathrm{Diff}_0(\Sigma_g)) = 0$

(See [EE]), so for a fixed genus g ($g \geq 2$) Lefschetz fibration over $\mathbb{D}^2$, its extension to a Lefschetz fibration over $\mathbb{S}^2$ is unique up to isomorphism.

The discussion in this section can be summarized by the following theorem:

Theorem 3.3. *Given the following:*

1. *an admissible sequence of vanishing cycles* $(\gamma_1, \gamma_2, \ldots, \gamma_\mu)$ *in* Σ_g,
2. *a homotopy class of loops* $\theta \mapsto \alpha_\theta$ *in* $\mathrm{Diff}_0(\Sigma_g)$,

one may construct a smooth manifold X *and* $\pi : X \to \mathbb{S}^2$ *by the procedure discussed such that* π *is a genus* g *Lefschetz fibration. Conversely, for any Lefschetz fibration* $\pi : X \to \mathbb{S}^2$, *there exists a sequence of Dehn twists in* Σ_g *and a homotopy class of loops* $\theta \mapsto \alpha_\theta$ *in* $\mathrm{Diff}_0(\Sigma_g)$ *such that* π *can be constructed by the same procedure.*

Proof. Given a sequence of vanishing cycles as in (1), Theorem 3.1 explicitly determines the framings for attaching the μ 2-handles to the boundary of $\mathbb{D}^2 \times \Sigma_g$ along the vanishing cycles corresponding to the Dehn twists. The boundary of the resulting manifold N fibers over $\mathbb{S}^1$, and since $\delta_{\gamma_\mu} \circ \cdots \circ \delta_{\gamma_2} \circ \delta_{\gamma_1}$ is isotopic to identity, $\partial N \cong \mathbb{S}^1 \times \Sigma_g$. Then the given homotopy class of loops in $\mathrm{Diff}_0(\Sigma_g)$ determines an attaching map, up to isotopy, to close up N with a copy of $\mathbb{D}^2 \times \Sigma_g$. Conversely, given a Lefschetz fibration π over $\mathbb{S}^2$ with μ critical points, one may draw a large closed disk D in $\mathbb{S}^2$ which encloses all critical values in its interior. Then $\pi^{-1}(D)$ can be seen as $\mathbb{D}^2 \times \Sigma_g$ attached with μ 2-handles. The attaching maps correspond to vanishing cycles by Theorem 3.1. Fact 3.1 asserts that $\delta_{\gamma_\mu} \circ \cdots \circ \delta_{\gamma_2} \circ \delta_{\gamma_1}$ is isotopic to the identity map. Finally the gluing map between $\pi^{-1}(D)$ and $\pi^{-1}(\overline{\mathbb{S}^2 - D})$ naturally gives a loop in $\mathrm{Diff}_0(\Sigma_g)$. $\diamond$ $\square$

This theorem immediately leads to an important consequence: all genus g Lefschetz fibrations can be characterized by admissible tuples of vanishing cycles (for $g \geq 2$). We will also see in the next section that all genus 1 Lefschetz fibrations of interest (i.e. relatively minimal and having at least one critical point) can also be represented by such a tuple.

It has been mentioned earlier that two distinct μ-tuples of vanishing cycles may correspond to isomorphic Lefschetz fibrations due to the choices made when a μ-tuple is formed. The corresponding changes for the μ-tuple of Dehn twists is as follows:

1. *A change of common identification.* More precisely, one may alter the identification $\pi^{-1}(a) \cong \Sigma_g$ by $\psi \in \text{Diff}(\Sigma_g)$. This leads to a change of the μ-tuple of Dehn twists from $(\delta_{\gamma_1}, \delta_{\gamma_2}, \ldots, \delta_{\gamma_\mu})$ to $(\delta_{\psi(\gamma_1)}, \delta_{\psi(\gamma_2)}, \ldots, \delta_{\psi(\gamma_\mu)}) = (\psi \circ \delta_{\gamma_1} \circ \psi^{-1}, \psi \circ \delta_{\gamma_2} \circ \psi^{-1}, \ldots, \psi \circ \delta_{\gamma_\mu} \circ \psi^{-1})$, i.e. by simultaneous conjugation.
2. *A change of order of the critical values.* Any change of order can be achieved by interchanging adjacent critical values for finitely many steps. Consider a switch of order of 2 adjacent critical values as shown in Figure 3.3. In the new picture, the identification is done along the arcs $\tilde{s}_i$ instead of s_i. Therefore the i-th term of the new tuple is just a_{i+1}, and the $i+1$-st term is a conjugate of a_i by a_{i+1}. A change of the μ-tuple from $(\delta_{\gamma_1}, \delta_{\gamma_2}, \ldots, \delta_{\gamma_{i-1}}, \delta_{\gamma_i}, \delta_{\gamma_{i+1}}, \ldots, \delta_{\gamma_\mu})$ to $(\delta_{\gamma_1}, \delta_{\gamma_2}, \ldots, \delta_{\gamma_{i-1}}, \delta_{\gamma_{i+1}}, \delta_{\gamma_{i+1}} \circ \delta_{\gamma_i} \circ \delta_{\gamma_{i+1}}^{-1}, \ldots, \delta_{\gamma_\mu})$ results. Such a change of the tuple is called a *Hurwitz move.*

The second change turns out to be more interesting.

Definition 3.9. *Two μ-tuples τ and σ of Dehn twists are said to be Hurwitz equivalent if τ can be changed to σ by a finite sequence of Hurwitz moves.*

Therefore, the classification of genus g Lefschetz fibration is reduced to the classification of admissible μ-tuples of Dehn twists of Σ_g up to Hurwitz equivalence and simultaneous conjugation, and the study of $\pi_1(\text{Diff}(\Sigma_g))$. Proceeding

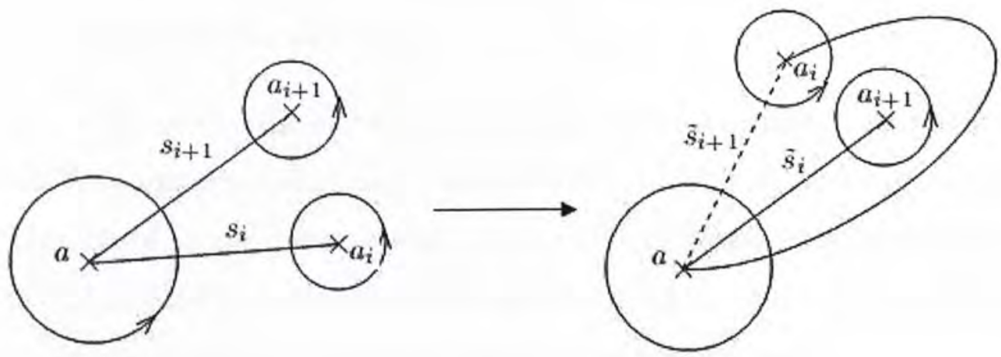

Figure 3.3: A Hurwitz move.

in this direction, the genus 1 case was completely solved by Moishezon using algebraic methods (Section 3.3), and the higher genus case was partially solved by Auroux (Section 3.4).

3.3 Genus 1

First we note some differences between the classification problems for the cases $g = 1$ and $g \geq 2$.

- For $g = 1$, all the neighborhoods of singular fibers are diffeomorphic, since every non-nullhomotopic vanishing cycle is non-separating, and for any two such vanishing cycles, there is a diffeomorphism of $\Sigma_1 \cong \mathbb{T}^2$ carrying one to the other. On the other hand, as g increases, there are more diffeomorphic types of neighborhoods of singular fibers and $\mathrm{MCG}(\Sigma_g)$ quickly gets more complicated. Therefore one expects that an increase in g causes more complication in classifying Lefschetz fibration over $\mathbb{D}^2$ due to more complications in the μ-tuple.

- However, it has been shown in [EE] that $\pi_1(\mathrm{Diff}_0(\Sigma_g))$ is isomorphic to

$\mathbb{Z} \oplus \mathbb{Z}$ for $g = 1$, but is trivial for $g \geq 2$. Thus a Lefschetz fibration over $\mathbb{D}^2$ extends uniquely over $\mathbb{S}^2$ for $g \geq 2$, but not so for $g = 1$.

Therefore, for $g = 1$, one may hope for a classification theorem which is as simple as only depending on the number of critical values of π, but must expect to face the obstacle of having different ways to close up a Lefschetz fibration over $\mathbb{D}^2$. It turns out that the obstacle can be eliminated just by the mild assumptions that the Lefschetz fibration is relatively minimal and has at least one critical point. This is a classical result due to Moishezon ([Mo]) and is the focus of this section.

We start with the following definition.

Definition 3.10. *Let $\pi : X \to \Sigma$ be a genus 1 Lefschetz fibration, $A = \{a_1, a_2, \ldots, a_\mu\}$ be the set of critical values of π, $a \in \Sigma - A$, and D_a be a small closed disk in* $\mathrm{Int}(\Sigma - A)$. *Let $\tau : \pi^{-1}(D_a) \to D_a \times \mathbb{T}^2$ be a trivialization of $\pi^{-1}(D_a)$. With this identification, for a loop $\alpha : \mathbb{S}^1 \to \mathrm{Diff}(\mathbb{T}^2)$ in $\mathrm{Diff}(\mathbb{T}^2)$, we have the diffeomorphism $\tilde{\alpha} : \pi^{-1}(\partial D_a) \to \partial(D_a \times \mathbb{T}^2)$ defined as $(\theta, x) \mapsto (\theta, \alpha_\theta(x))$. Then the Lefschetz fibration obtained from $\pi : X \to \Sigma$ by α-twisting at the point a is defined to be $\pi_{a,\alpha} : X_{a,\alpha} \to \Sigma$, where*

$$X_{a,\alpha} = \overline{X - \pi^{-1}(D_a)} \bigcup_{\tilde{\alpha}} (D_a \times \mathbb{T}^2),$$

$$\pi_{a,\alpha}(x) = \begin{cases} \pi(x) & \text{if } x \in \overline{X - \pi^{-1}(D_a)} \\ y & \text{if } x = (y, z) \in D_a \times \mathbb{T}^2 \end{cases}.$$

Intuitively, an α-twisting of a Lefschetz fibration is obtained by removing a neighborhood of a regular fiber, twisting its boundary by α and then gluing it back; this also corresponds to different ways of closing up the manifold $X_{1+\epsilon}$. We may also assume, without loss of generality, that $\alpha \in \mathrm{Diff}_0(\mathbb{T}^2)$.

The following lemma is the key which allows us to know when two Lefschetz fibrations resulted from non-homotopic twists of $\pi : X \to \Sigma$ are actually isomorphic; that is, when do two ways of closing up a genus 1 Lefschetz fibration over $\mathbb{D}^2$ lead to isomorphic Lefschetz fibrations. Note that this lemma is trivial for $g \geq 2$.

Lemma 3.2. *Let $\pi : X \to \Sigma$ be a genus 1 Lefschetz fibration and $\pi_{a,\alpha} : X_{a,\alpha} \to \Sigma$ be some α-twisting of $\pi : X \to \Sigma$. Suppose that the canonical homomorphism*

$$\rho : \pi_1(\Sigma - A, a) \to \mathrm{Aut}_+(H_1(\pi^{-1}(a), \mathbb{Z})),$$

defined by $Z \mapsto (\xi(Z))_*$, where ξ is the monodromy representation, is an epimorphism. Then there exists an isomorphism $(\psi : X \to X_{a,\alpha}, \phi : \Sigma \to \Sigma)$ between π and $\pi_{a,\alpha}$ such that

(1) $\phi = \mathrm{Id}_\Sigma$,

(2) if $\tau_1 : \pi^{-1}(D_a) \to D_a \times \mathbb{T}^2$ is a trivialization used to construct the α-twisting and $\tau_2 : \pi^{-1}_{a,\alpha}(D_a) \to D_a \times \mathbb{T}^2$ is the trivialization obtained from the construction of the α-twisting, then $\psi|_{\pi^{-1}(D_a)} = \tau_2^{-1} \circ \tau_1$.

We recall a fact about $\pi_1(\mathrm{Diff}_0(\mathbb{T}^2))$. It was shown in [EE] that the embedding given by $i : \mathbb{T}^2 \to \mathrm{Diff}_0(\mathbb{T}^2)$, $y \mapsto i_y$, $i_y(x) = x + y$ is a homotopy equivalence. So we identify $\pi_1(\mathrm{Diff}_0(\mathbb{T}^2))$ with $\pi_1(\mathbb{T}^2, \mathbb{Z}) = H_1(\mathbb{T}^2, \mathbb{Z})$, and consider α as an element of $H_1(\pi^{-1}(a), \mathbb{Z})$. More explicitly, if $\alpha = (m, n) \in H_1(\mathbb{T}^2, \mathbb{Z})$, then α corresponds to the twist

$$\tilde{\alpha} : \mathbb{S}^1 \times \mathbb{T}^2 \to \mathbb{S}^1 \times \mathbb{T}^2, \quad (\theta, \beta_1, \beta_2) \mapsto (\theta, \beta_1 + m\theta, \beta_2 + n\theta).$$

Sketch of proof. :
Let $Z : [0,1] \to \Sigma - A$ be a loop with $Z(0) = Z(1) = a$. Since ρ is a homomorphism, we assume that Z turns around one critical value (say a_1) with positive orientation. Let Z_* denote $\rho(Z)$. We will show that the α-twisted and the $Z_*(\alpha)$-twisted Lefschetz fibrations are actually isomorphic. First observe that there is an isotopy $\phi_t : \Sigma \to \Sigma, t \in [0,1]$, such that

1. $\phi_t \equiv \mathrm{Id}$ outside D_{a_1},

2. $\phi_t(a) = Z(t)$,

3. $\phi_1(D_a) = D_a$.

Thus D_a is carried around a_1 once with positive orientation given by Z. Then there is a key observation that there is an isomorphism (ψ', ϕ') between $\pi_{a,\alpha}$ and $\pi_{a,Z_*(\alpha)}$ such that $\phi' = \phi_1$. Intuitively, the isotopy ϕ_t can be lifted to $X_{a,\alpha}$, and this results in an extra Z_* acting on the twist α as the disk D_a goes around a_1 once.

Let e_1, e_2 be any free basis of $H_1(\pi^{-1}(a), \mathbb{Z})$. We observe that $\pi_{a,\alpha} = (\pi_{a,\alpha_1})_{a,\alpha_2}$ for $\alpha = \alpha_1 + \alpha_2$, so we only need to prove the lemma for $\alpha = e_1$ and $\alpha = e_2$. We assume $\alpha = e_1$; the proof for the other case is similar. By surjectivity of ρ, we choose $\Theta_1, \Theta_2 \in \mathrm{Aut}_+(H_1(\pi^{-1}(a), \mathbb{Z}))$ such that $\Theta_1(e_1) = (e_1 + e_2), \Theta_2(e_2) = -e_1$, and $Z_1, Z_2 \in \pi_1(\Sigma - A)$ such that $(Z_1)_* = \Theta_1, (Z_2)_* = \Theta_2$. Applying the preceding observation to (Z_1) and (Z_2), we obtain a sequence of isomorphisms

$$\pi_{a,e_1} \simeq \pi_{a,e_1+e_2} \simeq (\pi_{a,e_1})_{a,e_2} \simeq (\pi_{a,e_1})_{a,-e_1} \simeq \pi_{a,0} \simeq \pi.$$

(ψ', ϕ') can be chosen such that the final isomorphism between π_{a,e_1} and π satisfies properties (1) and (2). The proof is omitted. $\square$

<u>Remark</u>: The assumption that ρ is an epimorphism seems to restrict the application of this lemma to certain genus 1 Lefschetz fibrations. However, it can be shown that this assumption holds provided that the Lefschetz fibration has at least one critical point. Indeed, a Lefschetz fibration without critical points is just a fiber bundle; thus the lemma has already covered the most interesting case.

Now consider two genus 1 Lefschetz fibrations $\pi_1 : X_1 \to \Sigma$ and $\pi_2 : X_2 \to \Sigma$ with the same set of critical values $A = \{a_1, \dots, a_\mu\} \neq \emptyset$. Let $a \in \Sigma - A$.

Suppose we take an identification $\pi_1^{-1}(a) = \pi_2^{-1}(a)$ and denote the corresponding canonical homomorphisms by ρ_1 and ρ_2. Now let $Y \in \pi_1(\Sigma - A)$ such that $\rho_1(Y) = \rho_2(Y)$, and assume that $\rho_1(Y) = (\delta_{\gamma_1})_*, \rho_2(Y) = (\delta_{\gamma_2})_*$. We claim that γ_1 and γ_2 are homotopic. First note that the mapping class group of $\mathbb{T}^2$ is just $SL_2(\mathbb{Z}) = \mathrm{Aut}_+(H_1(\Sigma_g, \mathbb{Z}))$, with the isomorphism between them given by $\phi \mapsto \phi_*$. Thus δ_{γ_1} and δ_{γ_2} are homotopic. Also, with the classical Picard-Lefschetz formula:

Fact 3.2. *Let γ be a closed curve on Σ_g and δ be the right-handed Dehn twist about γ. Then $\delta_* : H_1(\Sigma_g, \mathbb{Z}) \to H_1(\Sigma_g, \mathbb{Z})$ is given by $z \mapsto z - (z \cdot \gamma)\gamma$,*

one can easily check that $\gamma_1 = \gamma_2 \in H_1(\Sigma_g, \mathbb{Z})$. When $g = 1$, this implies that γ_1 and γ_2 are homotopic. Therefore, if ρ_1 and ρ_2 coincide, then π_1 and π_2 actually have the same vanishing cycles. Moishezon gave an explicit construction and induction argument showing that the Lefschetz fibrations $\pi_1|_{\pi_1^{-1}(\mathbb{D}^2)}$ and $\pi_2|_{\pi_2^{-1}(\mathbb{D}^2)}$ are isomorphic, where $\mathbb{D}^2$ is a large disk containing A in its interior, and put that down as a lemma, though it also directly follows from the point of view of Kas' result. Therefore, π_1 and π_2 only differ by a twist. If ρ_1 and ρ_2 are epimorphisms too, then we may apply 3.2 to conclude that π_1 and π_2 are actually isomorphic. We have actually sketched the proof of the following lemma appeared in Moishezon's work:

Lemma 3.3. *Using the above notation, assume that ρ_1 and ρ_2 coincide and are epimorphisms. Then there exists an isomorphism of Lefschetz fibrations (ψ, ϕ) between π_1 and π_2 such that $\phi(a) = a$, ϕ induces identity on $\pi_1(\Sigma - A)$ and $\psi|_{\pi_1^{-1}(a)}$ is the identity map.*

From the above discussion, considering the μ-tuple $(\delta_{\gamma_1}, \delta_{\gamma_2}, \ldots, \delta_{\gamma_\mu})$ of Dehn twists of a Lefschetz fibration is equivalent to considering the μ-tuple of matrices $((\delta_{\gamma_1})_*, (\delta_{\gamma_2})_*, \ldots, (\delta_{\gamma_\mu})_*)$. We say that two μ-tuples $(\Theta_1, \Theta_2, \ldots, \Theta_\mu)$ and

$(\Theta'_1, \Theta'_2, \ldots, \Theta'_\mu)$ of matrices in $\mathrm{SL}_2(\mathbb{Z})$ which arise from twists are Hurwitz equivalent if the corresponding μ-tuples of Dehn twists are Hurwitz equivalent (cf. Section 3.2).

We now look more closely at the necessary conditions for a μ-tuple of matrices $(\Theta_1, \Theta_2, \ldots, \Theta_\mu)$ in $\mathrm{SL}_2(\mathbb{Z})$ to arise from an admissible μ-tuple of Dehn twists, which will lead to a classification of equivalence classes of such tuples.

(I) Since $\delta_{\gamma_\mu} \circ \ldots \circ \delta_{\gamma_1} = \mathrm{Id}$, it is obvious that $\Theta_\mu \cdot \Theta_{\mu-1} \cdots \Theta_1 = I_2$, where I_2 denotes the 2×2 identity matrix.

(II) Fix two generators e_1, e_2 on $H_1(\mathbb{T}^2, \mathbb{Z})$. Every non-trivial vanishing cycle γ on $\mathbb{T}^2$ is non-separating, so there exists an orientation-preserving diffeomorphism $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ which carries γ to e_1. The property of Dehn twists then tells us that $\delta_\gamma = \delta_{\phi(e_1)} = \phi^{-1} \circ \delta_{e_1} \circ \phi$, which implies that $(\delta_\gamma)_*$ and $(\delta_{e_1})_*$ are conjugate to each other. Since $(\delta_{e_1})_* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it follows that any matrix in $\mathrm{SL}_2(\mathbb{Z})$ which corresponds to a Dehn twist must be conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Therefore, the classification of genus 1 Lefschetz fibrations is reduced to the completely algebraic problem of classifying equivalence classes of μ-tuples of matrices which satisfies (I) and (II). We have the following powerful lemma:

Lemma 3.4. *Let* $\Theta' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $\Theta'' = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. *Any μ-tuple* $(\Theta_1, \Theta_2, \ldots, \Theta_\mu)$ *in* $\mathrm{SL}_2(\mathbb{Z})$ *satisfying* (I) *and* (II) *must satisfy* $\mu \equiv 0 \pmod 2$ *and is Hurwitz equivalent to the μ-tuple*

$$(\Theta', \Theta'', \Theta', \Theta'', \ldots, \Theta', \Theta'').$$

In particular, μ must be a multiple of 12.

The proof of the lemma is purely algebraic and relies on a theorem by Livne (cf. [Mo]). To see that μ is a multiple of 12, note that $(\Theta'' \cdot \Theta')$ is of order 6 in $\mathrm{SL}_2(\mathbb{Z})$. The lemma can be interpreted as follows: any relatively minimal genus 1 Lefschetz fibrations with μ singular fibers can be seen as the standard form $(\Theta', \Theta'', \Theta', \Theta'', \ldots, \Theta', \Theta'')$ with a suitable order of critical values, and as a consequence, such Lefschetz fibrations can be classified solely by their number of singular fibers.

Corollary 3.1. *Let $\pi : X \to \Sigma$ be a relatively minimal genus 1 Lefschetz fibration. If π has at least one critical point, then its canonical homomorphism ρ is an epimorphism.*

Proof. Let $(\delta_{\gamma_1}, \delta_{\gamma_2}, \ldots, \delta_{\gamma_\mu})$ be some monodromy representation of π. Lemma 3.4 implies that the μ-tuple of matrices corresponding to π is equivalent to $(\Theta', \Theta'', \Theta', \Theta'', \ldots, \Theta', \Theta'')$. Thus $\Theta', \Theta'' \in \rho(\pi_1(\Sigma - A))$. Since $\{\Theta', \Theta''\}$ generates $\mathrm{Aut}_+(H_1(\mathbb{T}^2, \mathbb{Z}))$, the result follows. □

The next question one may ask is: *For any μ which is a multiple of 12, does there exist a genus 1 Lefschetz fibration with μ singular fibers?* The answer can be reduced to constructing a genus 1 Lefschetz fibration π_0 with 12 singular fibers, since then one may take fiber sums of π_0 itself. In fact, π_0 can be obtained by constructing a Lefschetz pencil of cubic curves in $\mathbb{CP}^2$ and blowing up at the nine base points. It is well-known that π_0 has 12 singular fibers and its regular fiber is of genus 1 (cf. [GS], Lemma 3.1.4).

We conclude the discussion in this section by the theorem of the classification of genus 1 Lefschetz fibration:

Theorem 3.4. *Let $\pi : X \to \mathbb{S}^2$ be a relatively minimal genus 1 Lefschetz fibration with μ critical points ($\mu \neq 0$). Then $\mu \equiv 0 \pmod{12}$, and π is isomorphic to* *the fiber sum of $\frac{\mu}{12}$ copies of π_0, where π_0 is the Lefschetz fibration obtained from blowing up the nine base points of a Lefschetz pencil of cubic curves in $\mathbb{CP}^2$.*

Proof. Since π and the fiber sum of $\frac{\mu}{12}$ copies of π_0 are both genus 1 Lefschetz fibrations with μ critical points, it follows from Lemma 3.4 that their monodromy representations coincide for suitable choices of order of critical values. By Corollary 3.1, the canonical homomorphisms of both Lefschetz fibrations are epimorphisms. Thus it follows from Lemma 3.3 that the two Lefschetz fibrations are isomorphic. □

3.4 Genus 2

We have seen that the classification of genus 1 Lefschetz fibrations is complete; precisely, genus 1 Lefschetz fibrations can be exhausted by taking fiber sums of a holomorphic Lefschetz fibration. However, the situation quickly gets more complicated as the genus of the Lefschetz fibration increases. For $g = 2$, it is hard to completely classify factorization of positive Dehn twists as in the previous section because $\mathrm{MCG}(\Sigma_2)$ is already complicated.

However, the most natural way to tackle this problem is still to start with possible connections with holomorphic fibrations as hinted by $g = 1$ (we will see that the knowledge in holomorphic fibrations helps tackle the problem, at least in the genus 2 case). Thus the following questions arise:

1. Are all genus 2 Lefschetz fibrations holomorphic, i.e. isomorphic to a holomorphic Lefschetz fibration?

The answer is negative as shown in [OS]. By an explicit construction, there are infinitely many pairwise non-homeomorphic 4-manifolds which admit genus 2 Lefschetz fibrations but do not carry complex structures with either orientation.

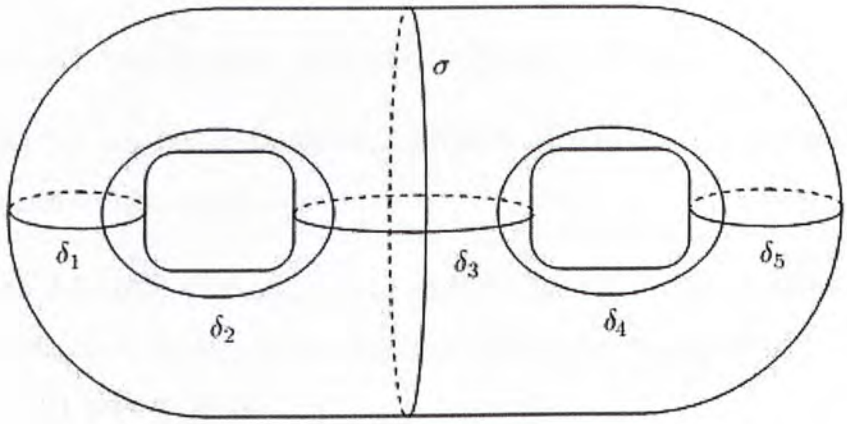

Figure 3.4: σ and the generators of $\mathrm{MCG}(\Sigma_2)$.

2. Are all genus 2 Lefschetz fibrations isomorphic to the fiber sum of holomorphic Lefschetz fibrations?

This question is still open. It was shown in [OS] that the answer is negative for infinitely many unspecified genera, and explicit examples for genus 3 were given by [Sm].

3. Does there exist a fixed genus 2 Lefschetz fibration L such that an arbitrary genus 2 Lefschetz fibration becomes holomorphic after taking fiber sum with a finite number of copies of L?

Auroux showed that the answer is affirmative in [A2]. The precise statement is given as follows. We define the generators $\delta_1, \delta_2, \dots, \delta_5$ of $\mathrm{MCG}(\Sigma_2)$ as shown in Figure 3.4. $\mathrm{MCG}(\Sigma_2)$ is generated by $\delta_1, \delta_2, \dots, \delta_5$ with relations as follows (cf. [B1]):

- $\delta_i \delta_j = \delta_j \delta_i$ for $|i-j| \geq 2$,
- $\delta_i \delta_{i+1} \delta_i = \delta_{i+1} \delta_i \delta_{i+1}$ for $1 \leq i \leq 4$,

- $(\delta_1\delta_2\delta_3\delta_4\delta_5)^6 = 1$,

- $I := \delta_1\delta_2\delta_3\delta_4\delta_5^2\delta_4\delta_3\delta_2\delta_1$ is central and satisfies $I^2 = 1$.

σ denotes the separating Dehn twist which separates Σ_2 into two genus 1 components and will be useful later.

Notation 3.2. *If $P = (p_1, p_2, \dots, p_k)$ and $Q = (q_1, q_2, \dots, q_l)$ are two sequences of Dehn twists, then (P, Q) denotes $(p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_l)$, and $(P)^m$ denotes $(P, P, \dots, P)$ with m terms.*

Theorem 3.5. *Let $W_0 = (T)^2, W_1 = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5)^6, W_2 = (\sigma, (\delta_3, \delta_4, \delta_5, \delta_2, \delta_3, \delta_4, \delta_1, \delta_2, \delta_3)^2, (T))$, where $T = (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_5, \delta_4, \delta_3, \delta_2, \delta_1)$. If F is any factorization of the identity element as a product of positive Dehn twists in $\mathrm{MCG}(\Sigma_2)$, then there exist integers $\epsilon \in \{0, 1\}, k \geq 0, m \geq 0$ and $N \in \mathbb{N}$ such that for any $n \geq N$, the factorization $(F, (W_0)^n)$ is Hurwitz equivalent to $((W_0)^{n+k}, (W_1)^\epsilon, (W_2)^m)$.*

From this theorem we can obtain the corollary:

Corollary 3.2. *Let $\pi : X \to \mathbb{S}^2$ be a genus 2 Lefschetz fibration. Then the fiber sum of π with sufficiently many copies of the rational genus 2 Lefschetz fibration with 20 irreducible singular fibers is isomorphic to a holomorphic fibration.*

The statement of Theorem 3.5 is purely algebraic, and in fact, so does the proof. However, that of Corollary 3.2 is geometric. Thus, to reach the corollary from the theorem, it is crucial to understand the geometric meaning of the Lefschetz fibrations corresponding to the factorizations defined in the theorem. More generally, we ask the following questions:

- Is there a geometric way to construct genus 2 Lefschetz fibrations?

- Can the monodromy factorizations of the Lefschetz fibrations resulted be computed?

- Does the construction cover all genus 2 Lefschetz fibrations?

It turns out that, by the use of *branched cover* and techniques in *braid theory*, one may directly construct Lefschetz fibrations with monodromy factorizations W_0 and W_1, and less directly, W_2. The following early result shows that holomorphic genus 2 Lefschetz fibrations without reducible fibers are well understood and can be seen as the origin of 3.5:

Theorem 3.6. *([H], [Ch]) Any relatively minimal genus 2 fibration $\pi : X \to \mathbb{S}^2$ (holomorphic with smooth total space, but not necessarily Lefschetz) with at least one critical point and without reducible fibers can be (holomorphically) deformed to a holomorphic Lefschetz fibration*

$$\pi' : X \xrightarrow{\tilde{\kappa}} \mathbb{F}_n := \mathbb{P}(O \oplus O(n)) \xrightarrow{p} \mathbb{S}^2,$$

where $\mathbb{P}(O \oplus O(n))$ is the Hirzebruch surface, $\tilde{\kappa}$ is a two-fold cover branched along a smooth curve $B \subset \mathbb{F}_n$ and such that one of the following holds:

(I_k) $n = 0, B \sim 6H + 2kF$ *is connected,* $k > 0$*;*

(II_k) $n = 1, B \sim 6H + 2kF$ *is connected,* $k \geq 0$*;*

(III_k) $n = 2k > 0, B = S_\infty \cup B'$ *with* $B' \sim 5H$ *connected,*

where H, S_∞, F denote a positive section, a negative section and a fiber of $\mathbb{F}_n$ respectively.

To understand this theorem, we begin with p. We consider an $\mathbb{S}^2$-bundle $\eta : P \to \mathbb{S}^2$, and let $B \subset P$ be a smoothly embedded surface which intersects a generic fiber F in 6 points, everywhere transverse except at a finite number of points with non-degenerate complex tangencies. Then we can consider the *braid monodromy* associated to B. First denote the generators of the braid group of $\mathbb{S}^2$ with respect to 6 points, $B(\mathbb{S}^2, 6)$, by $\zeta_1, \zeta_2, \ldots, \zeta_5$, the positive half-twists exchanging two consecutive points. Denote the tangency points by $z_1, z_2, \ldots, z_\mu$ and assume that η is injective on $\{z_1, z_2, \ldots, z_\mu\}$. For a positively oriented small circle C_i in $\mathbb{S}^2$ (started at x_i) around $\eta(z_i)$, there is a corresponding element in the braid group $B(\mathbb{S}^2, 6)$, moving the six points in $B \cap \eta^{-1}(x_i)$ around the tangency point. In this way we obtain a braid monodromy representation $\pi_1(\mathbb{S}^2, \{\eta(z_i)\}_{i\in I}) \mapsto B(\mathbb{S}^2, 6)$ which can be represented as a *braid factorization* $(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_\mu)$, each of ζ_i being positive half-twists of two points in $\mathbb{S}^2$.

It was shown by Siebert and Tian ([ST1]) that if $B \in H_2(P;\mathbb{Z})$ is divisible by 2, then one can construct a two-fold covering $\tilde{\kappa} : X \to P$ branched along B and get a genus 2 Lefschetz fibration $\tilde{\kappa} \circ \eta$. One imagines that $\tilde{\kappa}$ restricted to the preimage of a fiber of P without tangency points of B is a two-fold branched cover of Σ_g over $\mathbb{S}^2$, and the critical points of $\tilde{\kappa} \circ \eta$ correspond to the tangency points of B with the fibers of P.

The braid factorizations of the projections given in 3.6 can be computed by degenerating B into a singular curve with nodes and smoothing the nodes to generate the tangencies. The computation was sketched in Auroux's paper as a Lemma and was done in detail in [Ch], which showed the following results:

(I_k) $\mu = 20k, (\tau_1, \tau_2, \ldots, \tau_\mu) = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1)^{2k}$;

(II_k) $\mu = 20k+30, (\tau_1, \tau_2, \ldots, \tau_\mu) = ((\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1)^{2k}, (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)^6)$;

(III_k) $\mu = 40k, (\tau_1, \tau_2, \ldots, \tau_\mu) = (\zeta_1, \zeta_2, \zeta_3, \zeta_4)^{10k}$.

Secondly, to study κ, we look at the definition of two-fold branched cover: ([ST1]):

Definition 3.11. *A degree two map $\kappa : X \to Y$ of n-dimensional connected, oriented, differentiable manifolds whose set of critical values is a codimension* *two, oriented submanifold $B \subset Y$ is called a two-fold cover of Y with branch locus B if the following conditions are satisfied:*

1. $\kappa|_{X-\kappa^{-1}(B)} : X - \kappa^{-1}(B) \to Y - B$ *is a two-fold covering map,*

2. *locally along B, there exist oriented local coordinates $(x_1, x_2, \dots, x_n)$ of X and $(y_1, y_2, \dots, y_n)$ of Y such that*

$$\kappa : (x_1, x_2, \dots, x_n) \mapsto (y_1, y_2, \dots, y_n) = (x_1^2 + x_2^2, x_1 x_2, x_3, \dots, x_n).$$

For simplicity, we simply call κ a *cover branched along B* or just a *branched cover*. There are two particular cases which we are more interested in. When $X = \Sigma_g$ and $Y = \mathbb{S}^2$, B is exactly the finite set of critical values of κ and is one-to-one correspondent to $\kappa^{-1}(B)$. Applying Riemann-Hurwitz formula, one sees that B must have $2g + 2$ points. In particular, when $g = 2$, κ must have 6 branch points. And when X and Y are 4-manifolds, B is an embedded surface in Y.

Conversely, it was shown, independently by Siebert and Tian ([ST1]) and Fuller ([F1]) that all genus 2 Lefschetz fibrations can be factorized as a double covering of an $\mathbb{S}^2$-fibration and the projection of the fibration, and that the factorizations of the Lefschetz fibrations can be obtained by lifting the braid monodromy factorization through the map $\zeta_i \mapsto \delta_i$, that is, replacing ζ_i with δ_i; in fact, this construction holds for all *hyperelliptic Lefschetz fibrations*; see [ST1] for details.

Therefore, $n = 0, k = 1$ gives us the Lefschetz fibration with factorization W_0, and $n = 1, k = 0$ gives us that for W_1. Now, the holomorphicity theorem by [ST2] comes into play:

Theorem 3.7. *Any genus 2 transitive Lefschetz fibration without reducible fibers is holomorphic.*

Transitivity is a more technical assumption. For simplicity, we only state here as a fact that all factorizations which include all of $\delta_1, \delta_2, \ldots, \delta_5$ are transitive. We also call a Lefschetz fibration transitive if its factorization is transitive. With the discussion above, Theorem 3.7 can be reformulated as in [A1]:

Theorem 3.8. *Any transitive factorization of the identity element as a product of positive Dehn twists along non-separating curves in* $\mathrm{MCG}(\Sigma_2)$ *is Hurwitz equivalent to a factorization of the form* $((W_0)^k, (W_1)^\epsilon)$ *for some integers* $k \geq 0$ *and* $\epsilon \in \{0, 1\}$.

Therefore, Auroux's work is a generalization which deals with genus 2 Lefschetz fibrations with reducible fibers, which will be taken care by W_2.

One of the main parts of the proof of Corollary 3.2 is to construct the holomorphic fibrations with factorization W_2 and $((W_0)^k, (W_1)^\epsilon, (W_2)^m)$. We first look at W_2. Instead of directly considering double covers over smooth curves in $\mathbb{F}_1$ as in Theorem 3.6, we consider an algebraic curve $B_2 \subset \mathbb{F}_1$ in the linear system $|6H + F|$ which presents two triple points in the same fiber F_0. Then after some algebro-geometric operations, one obtains a holomorphic genus 2 Lefschetz fibration with one reducible fiber, where the vanishing cycle separates Σ_2 into 2 genus 1 components. It can be shown that the factorization of this Lefschetz fibration is indeed W_2. This construction can be generalized to obtain the Lefschetz fibration of desired factorization. We state the precise result without proof:

Theorem 3.9. *Fix integers* $m \geq 0, \epsilon \in \{0, 1\}$ *and* $k \geq \frac{3}{2}m + 1$. *Then* $\mathbb{F}_{m+\epsilon}$ *contains a complex curve* $B_{k,\epsilon,m}$ *in the linear system* $|6S + (m + 2k)F|$, *where* S *is a section of square* $(m + \epsilon)$ *and* F *is a fiber.* $B_{k,\varepsilon,m}$ *has* $2m$ *triple points lying in* m *distinct fibers of* $\mathbb{F}_{m+\epsilon}$ *as its only singularities.*
After blowing up $\mathbb{F}_{m+\epsilon}$ *at the* $2m$ *triple points, passing to a double covering, and blowing down* m *rational* -1*-curves, we obtain a complex surface* $X_{k,\epsilon,m}$ *and a*

holomorphic genus 2 Lefschetz fibration $\pi_{k,\epsilon,m} : X_{k,\epsilon,m} \to \mathbb{S}^2$ with monodromy factorization $((W_0)^k, (W_1)^\epsilon, (W_2)^m)$.

The proof of Theorem 3.5 is algebraic. Let π be a genus 2 Lefschetz fibration with factorization F. If π has no separating Dehn twists, then F may be made transitive by attaching a copy of W_0. By applying Theorem 3.7, we get that $(F, (W_0))$ is Hurwitz equivalent to $((W_0)^k, (W_1)^\epsilon)$ for some $k \in \mathbb{N}$ and $\epsilon \in \{0, 1\}$, and in particular, holomorphicity follows. The case that π has separating Dehn twists follows from induction on the number of separating Dehn twists and involves some algebraic lemmas for moving the separating Dehn twists around.

To summarize the discussion in this section, we prove Corollary 3.2.

Proof of Corollary 3.2: Let F be a monodromy factorization corresponding to π. By Theorem 3.5, we know that $(F, (W_0)^n)$ is Hurwitz equivalent to $((W_0)^{n+k}$, $(W_1)^\epsilon, (W_2)^m)$ for any sufficiently large n. In particular we can choose n such that $n + k \geq \frac{3}{2}m + 1$, so that, according to Theorem 3.9, the Lefschetz fibration with factorization $((W_0)^{n+k}, (W_1)^\epsilon, (W_2)^m)$ is holomorphic. Observe that $(F, (W_0)^n)$ corresponds to attaching n copies of the Lefschetz fibration with factorization W_0 (call it π_0). By Theorem 3.6, π_0 is indeed the rational genus 2 Lefschetz fibration with 20 irreducible singular fibers. So we are done. □

3.5 Genus $g \geq 3$

There is another gap in the difficulty of the clssification of Lefschetz fibration when we move from $g = 2$ to $g \geq 3$ because the presentation of $\mathrm{MCG}(\Sigma_g)$ has an extra relation for $g \geq 3$ ([W]). It is still possible to carry on with the approaches mentioned in the previous section, but likely, there will be some price to pay.

It was mentioned in the previous section that a particular kind of Lefschetz fibrations, namely hyperelliptic Lefschetz fibrations. can be seen as double coverings of $\mathbb{S}^2$-fibrations. Hyperelliptic Lefschetz fibrations exhibit a special symmetry on Σ_g with respect to a *hyperelliptic structure*, which is, roughly speaking, a two-fold branched cover $\kappa : \Sigma_g \to \mathbb{S}^2$ along B with a *hyperelliptic involution* ι, which is the non-trivial automorphism of the 2-fold covering map $\kappa|_{\kappa^{-1}(B)}$.

The sense of symmetry is given as follows: a diffeomorphism $\phi : \Sigma_g \to \Sigma_g$ is said to be *symmetric* if $\iota \circ \phi = \phi \circ \iota$. The hyperelliptic mapping class group HMCG(Σ_g) is defined to be the subgroup of symmetric diffeomorphisms in MCG(Σ_g). A Lefschetz fibration is called a hyperelliptic Lefschetz fibration if it can be represented by a μ-tuple of Dehn twists in HMCG(Σ_g).

One of the formulations of the result in [ST1] is as follows:

Theorem 3.10. *For $g \geq 2$ there is a one-to-one correspondence between:*

- *Isomorphism classes of genus g Lefschetz fibrations with hyperelliptic structure ($\pi : X \to \mathbb{S}^2, \kappa : \pi^{-1}(s) \to \mathbb{S}^2$), s being a regular value of π), with μ singular fibers, t of which are reducible, and*

- *Isomorphism classes of pairs $(\tilde{P}, B)$ with $\tilde{P} \to \mathbb{S}^2$ being an $\mathbb{S}^2$-fibration with t fibers of type Γ_2 and $B \subset \tilde{P}$ a branch surface of type (g, μ, t).*

It was also shown in the same work that B is symplectic. Therefore, the classification of hyperelliptic Lefschetz fibrations is reduced to the classification of some symplectic submanifolds in rational ruled surfaces. Saying that B a branch surface of type (g, μ, t) roughly means that it satisfies some 2-divisibility conditions, and that $B \to \tilde{P} \to \mathbb{S}^2$ is a $(2g+2)$-fold branched cover, with the number of ordinary critical points determined by μ and t. In simpler terms, any hyperelliptic genus g Lefschetz fibration without reducible singular fibers can be seen as a

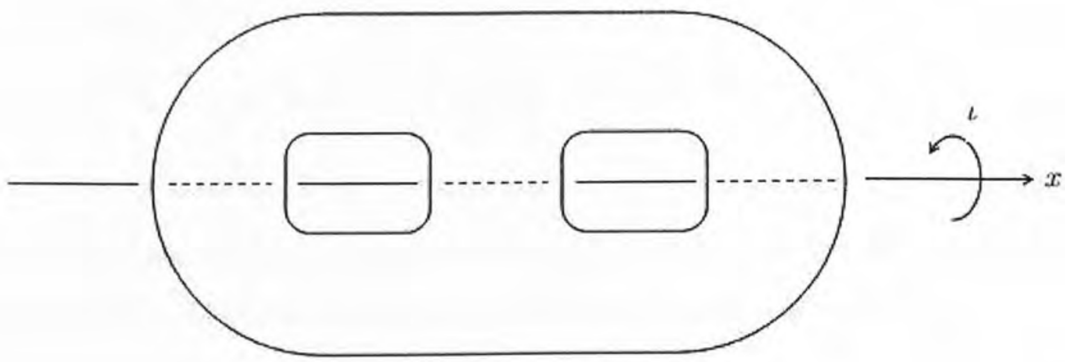

Figure 3.5: A standard hyperelliptic involution.

two-fold branched cover of an $\mathbb{S}^2$-bundle, and more generally, those with reducible singular fibers can be seen as a two-fold branched cover of an $\mathbb{S}^2$-fibration ([F1]).

We state a theorem which gives the presentation of $\mathrm{HMCG}(\Sigma_g)$, which implies that Theorem 3.10 is not applicable on all genus g Lefschetz fibrations for general g. In fact, there is a standard hyperelliptic structure on Σ_g. First embed Σ_g into $\mathbb{R}^3$ such that the central axis of Σ_g lies on the x-axis and that Σ_g is rotationally symmetric about the x-axis (Figure 3.5). Define $\iota : \Sigma_g \to \Sigma_g$ to be the $180°$ rotation about the central axis, and we assume the existence of a branched cover $\kappa : \Sigma_g \to \mathbb{S}^2$ such that ι is the hyperelliptic involution of κ. Therefore the critical points of κ are exactly the intersection of Σ_g and the central axis, and (κ, ι) gives a hyperelliptic structure on Σ_g. With this hyperelliptic structure, Birman [B1] showed that $\mathrm{HMCG}(\Sigma_g)$ is a $\mathbb{Z}/2\mathbb{Z}$-extension of $\mathrm{MCG}(\mathbb{S}^2, 2g+2)$, the mapping class group of $\mathbb{S}^2$ with $2g+2$ marked points, and can be seen as a subgroup of $\mathrm{MCG}(\mathbb{S}^2, 2g+2)$ with generators $\delta_1, \delta_2, \ldots, \delta_{2g+1}$ with the following relations ([B2]):

- $\delta_i\delta_j = \delta_j\delta_i$ for $|i-j| \geq 2$;

- $\delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1}$ for $1 \leq i \leq 2g$;

- $I := \delta_1\delta_2 \dots \delta_{2g+1}\delta_{2g+1} \dots \delta_2\delta_1$ satisfies $Ix_i = x_iI$, $I^2 = 1$;

- $(\delta_1\delta_2 \dots \delta_{2g+1})^{2g+2} = 1$.

Indeed, this presentation coincides with $\mathrm{MCG}(\Sigma_g)$ for $g = 2$, but not for $g \geq 3$, however ([W]). Therefore every genus 2 Lefschetz fibration is hyperelliptic, but the same does not hold for higher genus Lefschetz fibrations.

Next we state another theorem by Auroux which deals with classifying genus g Lefschetz fibration *up to fiber sum stabilization*. Precisely, the following question is answered: For each genus g, does there exist a standard Lefschetz fibration π_g^0 such that for any two Lefschetz fibrations π and π', $\pi\#n\pi_g^0$ and $\pi'\#n\pi_g^0$, the fiber sums of π and π' with n copies of π_g^0, become isomorphic for sufficiently large n?

An obvious necessary condition is that π and π' have the same number of singular fibers of each type. With this assumption, a positive answer has been shown for $g = 1$ (taking π_1^0 as the one constructed at the end of Section 3.3) and for $g = 2$, as an easy consequence of Theorem 3.5. For $g \geq 3$, Auroux ([A1]) gave an affirmative answer, but under some extra conditions:

Theorem 3.11. *For every g there exists a genus g Lefschetz fibration $X_g^0 : \pi_g^0 \to \mathbb{S}^2$ such that for any two genus g Lefschetz fibrations $X_g : \pi \to \mathbb{S}^2$ and $X_g' : \pi' \to \mathbb{S}^2$ with the same number of singular fibers of each type, each equipped with a distinguished section, satisfying the following:*

1. *X_g and X_g' have the same Euler characteristic and signature,*

2. *the distinguished sections of π and π' have the same self-intersection,*

then there exists $N \in \mathbb{N}$ *such that* $\pi \# n\pi_g^0$ *and* $\pi' \# n\pi_g^0$ *are isomorphic for any* $n \geq N$.

The requirement that X_g and X'_g have the same Euler characteristic is equivalent to π and π' having the same number of singular fibers. The signature assumption and condition (2) are well-known to hold for hyperelliptic Lefschetz fibrations, and genus 2 Lefschetz fibrations in particular, as a consequence of having the same number of singular fibers of each type.

Theorem 3.11, together with Donaldson's existence theorem of Lefschetz pencils on compact symplectic 4-manifolds (Theorem 2.3), lead to a classification of compact symplectic 4-manifolds up to symplectic blow-ups and symplectic sums:

Corollary 3.3. *Let* X, X' *be two integral compact symplectic 4-manifolds with the same* $(c_1^2, c_2, c_1 \cdot [\omega], [\omega]^2)$. *Then there exists* g *such that* X *and* X' *become symplectomorphic after sufficiently many blow-ups and symplectic sums with* X_g^0.

This is proved by considering Lefschetz pencils of curves of the same degree on X and X'. Then the Lefschetz pencils can be blown up at the base points to get two Lefschetz fibrations of the same genus. Then Theorem 3.11 can be applied.

Bibliography

[A1] D. Auroux, *A stable classification of Lefscehtz fibrations*, Geometry and Topology 9 (2005), 203-217.

[A2] D. Auroux, *Fiber sums of genus 2 Lefschetz fibrations*, Proceedings of 9^{th} Gokova Geometry-Topology Conference, pp 1-10.

[B1] J. Birman, *Braids, links and mapping class groups*, Princeton University Press 1974.

[B2] J. Birman and H. Hilden, *On the mapping class groups of closed surfaces as covering spaces*, in: *Advances in the theory of Riemann surfaces* (L. Ahlfors et al. eds.), Ann. Math. Studies 66 (1971), 81-115.

[Ch] K. Chakiris, *The monodromy of genus two pencils*, Thesis, Columbia University, 1983.

[D1] S. K. Donaldson, *Lefschetz pencils on symplectic manifolds*, J. Differential Geometry 53 (1999), 205-236.

[D2] S. K. Donaldson, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geometry 44 (1996), 666-705.

[EE] C. J. Earle and J. Eells, *The diffeomorphism group of a compact Riemann surface*, Bull. Amer. Math. Soc., 73 No. 4 (1967), 557-560.

[F1] T. Fuller, *Hyperelliptic Lefschetz fibrations and branched covering spaces*, Pacific J. Math. 196 (2000), 369-393.

[F2] T. Fuller, *Lefschetz Fibrations of 4-Dimensional Manifolds*, Cubo Mat. Educ. Volume 5 (2003), 275-294.

[GS] R. Gompf and A. Stipsicz, *4-manifolds and Kirby calculus*, Grad. Studies in Math. 20, Amer. Math. Soc., Providence, 1999.

[H] E. Horikawa, *Algebraic surfaces of general type with small* c_1^2, Ann. Math. 104 (1976) 357-387, II. Inv. Math. 37 (1976) 121-155, III. Inv. Math. 47 (1978) 209-248, IV. Inv. Math 50 (1979) 103-128.

[K] A. Kas, *On the Handlebody Decomposition Associated to a Lefschetz Fibration*, Pacific J. Math. 89 (1980), 89-104.

[McS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, 1995.

[Mo] B. Moishezon, *Complex Surfaces and Connected Sums of Complex Projective Planes*, Lecture Notes in Mathematics, 603 Springer-Verlag, 1977.

[OS] B. Ozbagci, A. Stipsicz, *Noncomplex smooth 4-manifolds with genus 2 Lefschetz fibrations*, Proc. Amer. Math. Soc. 128 (2000), 3125-3128.

[Sm] I. Smith, *Lefschetz pencils and divisors in moduli space*, Geometry and Topology 5 (2001), 579-608.

[Ste] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, 1951.

[ST1] B. Siebert and G. Tian, *On hyperelliptic* $C^\infty-$*Lefschetz fibrations of four manifolds*, Commun. Contemp. Math. 1 (1999), 255-280.

[ST2] B. Siebert, G. Tian, *On the holomorphicity of genus two Lefschetz fibrations*, Ann. Math., 161 (2005), 959-1020.

[Sti] A. Stipsicz, *Indecomposibility of certain Lefschetz Fibrations*, Proc. Amer. Math. Soc. 129 (2000), 1499-1502.

[Th] W. Thurston, *Some simple examples of symplectic manifolds*, Proc. Amer. math. Soc. 55 (1976), 467-468.

[W] B. Wajnryb, *A simple presentation for the mapping class group of an orientable surface*, Israel Journal of Mathematics 45 (1983), 157-174.